SKRIPTUM

Einführung in die Superstring–Theorie

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Chapter 1

Introduction

As it became clear that general relativity and Maxwell theory are both intimately tied to the concept of local symmetries, a unified description of the forces of nature became conceivable. In general relativity (GR), a local choice of coordinates has to be made and the action is a local functional of the metric and of the matter fields that is independent of this choice. Electrodynamics (ED) can be described very efficiently by a vector potential $A^\mu = (\phi, \vec{A})$, with the field strengths $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ being invariant under gauge transformations $\delta A_\mu = \partial_\mu \Lambda(x^\mu)$ [H. Weyl, 1918]. While classical physics can be formulated in terms of the field strengths, a local description of the coupling to quantum mechanical wave functions requires the gauge potentials, as is illustated by the Aharonov–Bohm effect [ah59].

A promising framework for unification was suggested by Kaluza and Klein [ka21]: They proposed that space-time is 5-dimensional, but with only 4 approximately flat directions and with one direction curled up on a small circle. Then the off-diagonal entry of the metric $A_\mu := g_{4\mu}$ transforms as a vector from the 4-dimensional point of view and serves as the gauge potential. It is still unknown, however, how the dynamics of the gravitational field generates the vacuum expectation value (VEV) for the metric. In the 20s it was not even possible to pose this question within any proper framework, which certainly has to incorporate a quantum mechanical treatment of the gravitational interactions.

After a long development of quantum field theory, techniques for a perturbative analysis of quantum gravity became available in the 60s [fe63, De65]. At about the same time the standard model of strong and electroweak particle interations, an $SU_3 \times SU_2 \times U_1$ gauge theory that is spontaneously broken to $SU_3 \times U_1$ below $100GeV$, was constructed. This led to the discovery of asymptotic freedom [co73] of QCD and – a decade later – to the detection of the W and Z bosons, which mediate weak interactions. Attempts at a group theoretical 'grand unification' of the standard model by a gauge theory with SU_5 [ge74] or even larger gauge groups [sl81] produced surprisingly good predictions for the ratio of the W and Z masses and, at the same time, lead to a unification scale of $10^{14} - 10^{16} GeV$, far above the weak (Fermi) scale of $300 GeV$. Since quantum gravity is bound to become important at the Planck mass[1] $M_{Pl} = \sqrt{\hbar c/G_N} = 1.22 \times 10^{19} GeV/c^2$, this may be regarded as an indication that gravity should no longer be ignored in particle physics.

It turned out that a perturbative quantization of gravity is spoiled by non-renormalizability, i.e. an infinite number of divergent quantum corrections that cannot be controlled by symmetries. An important example of such a correction is the one that modifies the cosmological constant Λ (the energy density of the vacuum becomes observable in gravitational interactions). The experimental bound for the physical value is best characterized by the tiny dimensionless ratio $|\Lambda|/M_{Pl}^2 < 10^{-121}$ [data].[2] It is clear that such a tiny quantity should be explained by a symmetry, the best candidate for which is supersymmetry (SUSY) [WE83]: Note that the energy of a harmonic oscillator is $E = \frac{1}{2}\omega(a^\dagger a \pm aa^\dagger) = \omega(a^\dagger a \pm \frac{1}{2})$ for excitation modes that are quantized according to bosonic/fermionic statistics $a^\dagger a \mp aa^\dagger = 1$. The zero point energies are, therefore, of equal size and opposite sign. The energy operators for second quantized free fields consist of an infinite sum of such oscillator terms. In order to have a cancellation of zero point energies we should thus have an equal number of bosonic and fermionic degrees of freedom and a symmetry that controlls the cancellation when interactions are turned on. Due to the spin statistics theorem [ST64] a *physical* symmetry that transforms commuting into anti-commuting fields should be in a spin 1/2 representation of the Lorentz group and should be implemented by an anticommuting operator in Hilbert space.

Because the anti-commutator of two SUSY transformations generates translations, the local (or gauged) version or supersymmetry automatically contains gravity and is hence called supergravity (SUGRA). In the late 70s and early 80s it was hoped that SUGRA might cure the divergences of quantum gravity. This also lead to a revival of the ideas of Kaluza and Klein, but now with a higher dimensional compactification space in order to be able to incorporate the whole standard model of particle interactions into a (super)geometrical picture. It was shown that the standard model can only be obtained from at least 11 dimensions, which, at the same time, is the maximal dimension allowed for supergravity (a Weyl fermion has $2^{d/2-1}$ components in d (even) dimensions while a massless vector field has $d-2$ transversal degrees of freedom; this leads to a mismatch for $d > 11$). But it is hard to get chiral fermions by starting in an odd number of dimensions [ba87]. The alternative of adding gauge symmetries to a 10-dimensional theory by hand goes against the original spirit of the ideas of Kaluza and Klein. Even worse, it turned out that SUGRA could not solve the problem with divergences.

[1] In *cgs* units $M_{Pl} = 2.2 \times 10^{-5} g$; the corresponding length scale is $\sqrt{\hbar G_N/c^3} = 1.6 \times 10^{-33} cm$.

[2] The experimental bound for the photon mass is $m_\gamma < 3 \times 10^{-33} MeV$, so that electromagnetic gauge invariance appears to be an exact symmetry of nature.

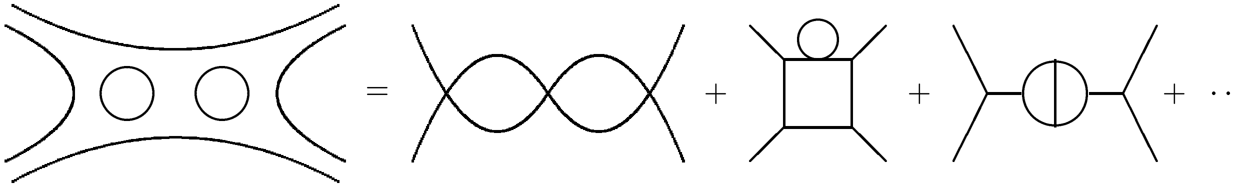

Fig. 1: $g = 2$ world sheet and some corresponding 2-loop Feynman graphs.

1.1 String unification

String unification apparently works in a rather different way: Here the fundamental object is a thread or a loop in space-time which, during time evolution, sweeps out a surface that is called **world sheet** (in analogy to the world line of a point particle). The dynamics is described by an action that is proportional to the area of that surface, and hence in purely geometrical terms. Particles are oscillation modes of the string, and interactions occur by joining and splitting of string configurations, as is shown in Fig. 1. This has two important consequences:

- There is no interaction point (the apparent splitting point changes under Lorentz transformations), which avoids the UV divergences of second quantized point particle theory.
- There is a unique geometrical interaction, which unifies an a priory infinite number of independent couplings among different fields.

Alltogether, string theory leads to a unification of interactions *and* to a unification of particles.

From a more modern point of view we may think of the world sheet as an independent two-dimensional space with local coordinates σ^m. The string coordinate functions $X^\mu(\sigma)$ are quantum fields on that space and describe its embedding into a **target space**, which may itself be a topologically non-trivial manifold with local coordinates X^μ. The geometrical description of the action ensures that is can be constructed in a coordinate invariant way as a sum over terms that are defined via local coordinates. For historical reasons such a quantum field theory is called a **$\boldsymbol{\sigma}$ model**. Unfortunately, there are two big problems with this approach:

- Scattering amplitudes are only defined as an infinite sum over different world sheet topologies, i.e. we do not have a non-perturbative definition of string theory. The sum over topologies may be badly divergent.
- To define a string theory we need to choose some fixed **background** target space geometry (or, in a more abstract description, a conformal field theory). Although different choices of this background should lead to equivalent physics, this is not manifest in the construction.

It may or may not be the case that some more elaborate version of string field theory [zw93], which is sometimes referred to by the name 'third quantization', will eventually solve this problem. At present it is not at all clear what string theory really is.

In any case, we do have a well-defined prescription for constructing a finite perturbative expansion of scattering amplitudes for the particles that effectively describe the physics of a string model at large distances (i.e. distances larger than $10^{-33}cm$). This is done in terms of 2-dimensional conformally invariant quantum field theories and a lot has been learned about how the properties of these world sheet models are related to the resulting space-time physics that we can proble in accelerator experiments or astrophysical observations. An incomplete dictionary is compiled at this point for later reference:

Space-time	$\longleftrightarrow$	**world sheet**
Einstein equations	$\longleftrightarrow$	conformal invariance
gauge invariance	$\longleftrightarrow$	current algebra (Kac–Moody)
anomaly cancellation	$\longleftrightarrow$	modular invariance
$N = 1$ supergravity	$\longleftrightarrow$	$N = 2$ supersymmetry
space-time geometry	$\searrow$	conformal field theory
particle spectrum	$\swarrow$	

In the last entry of this table it is indicated that a given background space-time geometry leads to a well-defined conformal field theory on the world sheet. Note, however, that this arrow cannot be reversed: In case of strong curvature quantum corrections can be large so that classically different background geometries can lead to quantum mechanically equivalent string theories (and therefore to the same space time physics).

It was already indicated above that we actually need a supersymmetric version of string theory. This has two reasons: The bosonic string has a tachyonic excitation in its spectrum, which indicates that it is unstable and which leads to IR divergences in perturbation theory. Furthermore, we want to describe spin 1/2 particles like electrons or nucleons, and these are missing in the excitation spectrum of the bosonic theory. This leads to the superstring whose conformal anomaly vanishes in 10 dimensions, which is also consistent with an effective low energy supergravity theory.

Actually, there seem to be 5 consistent supersymmetric string theories in 10 dimensions [GR87]: For a flat target space the coordinate fields satisfy the 2D wave equation $\Box X^\mu = (\partial_1 + \partial_0)(\partial_1 - \partial_0)X^\mu = 0$, whose general solution is a superposition of left-moving and right-moving excitations. In case of open superstrings, called type I, boundary conditions lead to a reflection of these modes at the string ends (the type I theory also contains closed string states in its spectrum since they can be formed by interactions, and its consistency requires to consider unoriented world sheets and Chan–Paton factors for the gauge group SO_{32} [GR87]). For oriented closed strings we have to make a choice in the relative chirality of the left and right moving supersymmetries. This leads to the type IIA and type IIB theories with $N = 2$

space-time SUSY, the latter of which is chiral. Moreover, in the closed string case we may even chose to combine a left-moving bosonic string with a right-moving fermionic string. The $D_{bos.} - D_{ferm.} = 26 - 10 = 16$ single left-moving bosons cannot be interpreted as space-time coordinates but rather show up as gauge degrees of freedom. This asymmetric construction is strongly constrained by potential quantum violation of symmetries (space-time anomalies coming from a violation of WS modular invariance), so that only two consistent choices exist: The heterotic strings with gauge groups $E_8 \times E_8$ and SO_{32}, respectively.

An important phenomenon in string theory (and many of its building blocks and effective theories) is duality, which means that different classical theories can lead to the identical quantum mechanical models. The oldest example of this type – except for bosonization – is the $R \longleftrightarrow 1/R$ duality of strings compactified on a circle with radius R. This duality exchanges winding modes and oscillation modes and is a stringy phenomenon that has no analogue in Kalaza–Klein compactification. Mirror symmetry is a generalization of this duality to certain 3-dimensional curved complex manifolds that can be used to construct more realistic models. In that case quantum mechanically equivalent backgrounds differ not only in size but also in shape and even topology, which leads to exciting implications for both, mathematics and physics [as94, mo95].

Quite recently, this duality business has even been extended to dualities among the above 5 different string theories, or rather their lower dimensional relatives which are continuously connected to 10-dimensional theories by letting some compactification radii go to infinity [as95, fe95, ka95, va95, wi95]. While most of these string-string dualities are still hypothetical, they already survived a number on non-trivial tests [$\mathtt{ka}_1$95] and they may well teach us some important lessons towards understanding what string theory really is. There are attempts to understand these dualities in terms of hypothetical 11- or 12-dimensional theories, called M and F theory, respectively [$\mathtt{wi}_1$96, va96, $\mathtt{be}_3$96, ma96].

The present lecture notes on strings are largely based on the books by Green, Schwarz and Witten [GR87] and by Lüst and Theisen [LU89]. There are many other good sources, like the book by Kaku [KA88] and the lecture notes by Kiritsis [Ki97], which can be obtained via internet. In particular I recommend the excellent books by Polchinski [PO98]. Most books on string theory use the sign convention $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$ for the Minkowski space metric, so that mass and momentum are related by $m^2 = -p^2$ (we use natural units $\hbar = c = 1$). This convention facilitates to keep equations consistent while performing the Wick rotation to Euclidean space. We will, however, use the convention $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$, which is mostly preferred in QFT textbooks, so that $t = x^0 = x_0$, which is somewhat nicer in the Hamilton formalism.

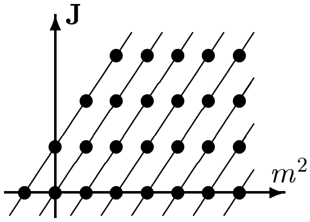

Fig. 2: Regge trajectories

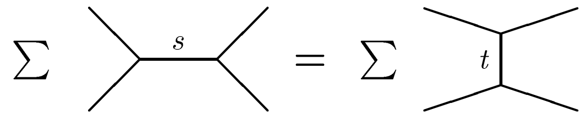

Fig. 3: Duality

1.2 History of string theory

String theory was discovered in the late 60s as a model for hadron resonances, large numbers of which were found with a spin–mass relation described by Regge trajectories $J = \alpha_0 + m^2\alpha'$, as shown in Fig. 2. Renormalizable QFTs, however, were and are known only for spin $J \in \{0, \frac{1}{2}, 1\}$: Scalars interacting by exchange of a spin J particle, for example, have an amplitude $A_J(s,t) \sim \frac{s^J}{t-m^2}$ where $s = (p_1+p_2)^2$, $t = (p_1+p_3)^2$ and $u = (p_1+p_4)^2$ are the Mandelstam variables[3] for scattering of two particles with momenta p_1 and p_2 to particles with outgoing momenta $-p_3$ and $-p_4$, because there are J derivatives in the interaction term[4] [GR87]. This generated doubts that hadron resonances were really fundamental particles.

At that time analytical properties of the S-matrix, like the relation between s and t channel amplitudes, were studied extensively, and the idea of duality was born [do68]. It states that s and t channel contributions should be equal, instead of being added as in QFT (see Fig. 3). This hypothesis had only marginal experimental support, but Veneziano [ve68] guessed an amplitude with the desired property, namely $A(s,t) + A(t,u) + A(u,s)$ with

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} = \int_0^1 dz\, z^{-1-\alpha(s)}(1-z)^{-1-\alpha(t)}, \qquad \alpha(s) = \alpha_0 + \alpha' s. \tag{1.1}$$

It has exponentially soft UV behaviour, whereas for QFTs cross sections only decrease like inverse powers, and it has infinitely many poles, i.e. describes infinitely many particles.

It turned out that this dynamics can be described by a string picture, with the observed particles being the excitation modes of the string. The Nambu-Goto action for the string is proportional to the area of the world sheet, just as the action for a relativistic point particle is proportional to its proper time $S[x] = -m\int ds = -m\int d\tau\,\sqrt{\dot{x}^2}$, where τ parametrizes the world line $x^\mu(\tau)$. From this geometrical picture of string interactions (see Fig. 1) duality is now apparent. Furthermore, the UV behaviour of string amplitudes is exponentially soft because there are no localizable interaction points on a smooth surface: The symmetries of the string organize the contributions of infinitely many massive particles of high spin in such a way that

[3] $s/t/u$ is the total energy squared in the rest frame of the $s/t/u$ chanel, and $s+t+u = \sum m_i^2$.

[4] Loop amplitudes $\sim \int dp^n A^2(p)/p^4$ are UV finite for $J < 1$ and have a potentially renormalizable logarithmic divergence for $J = 1$ in $n = 4$ dimensions.

the sum of an infinite number of terms with polynomial growth is exponentially small, like in the Taylor expansion of $\exp(-x)$.

In the early 70s QCD turned out to do better in describing hadron interactions[5] (asymptotic freedom in 1973, etc.). But Scherk and Schwarz showed that strings provide a promising theory for quantum gravity [sc74]: There always is a massless spin 2 excitation – the graviton – and there are no UV divergences, because there are no point-like interactions. The bad news, however, was that, in light-cone quantization [go73], Lorentz invariance is broken in $D \neq 26$, and that the intercept α_0 turned out to be positive so that the squared mass of the ground state is negative (tachyonic) and the theory is, at best, formulated in an unstable 'vacuum'.

This inconsistency was eventually cured by fermions, which had already been introduced into dual models by Ramond [ra71] and by Neveu and Schwarz [ne71] in order to describe fermionic hadron resonances. A generalization of the Nambu action to the 'spinning string' [br76, de76] was possible, however, only after some development of supersymmetry. Due to the additional fermionic degrees of freedom on the world sheet the critical dimension of the spinning string reduces to $D = 10$. But this model still is plagued by inconsistencies related to a tachyon, which eventually was thrown out by the GSO projection [gl76]. The resulting spectrum of states then turned out to be space-time supersymmetric, i.e. contains an equal number of bosonic and fermionic degrees of freedom, which are related by an anticommuting symmetry.[6] There is an alternative formulation, called the Green–Schwarz superstring [GR87], which is manifestly space-time supersymmetric. We will, however, mainly consider the RNS model with the *manifest* supersymmetry living on the world sheet.

After almost 10 years of underground development of string theory and many fruitless efforts to find a viable model for SUGRA Kaluza–Klein unification it was time for the string revolution, which came in 1984 with the discovery of the Green–Schwarz mechanism [gr84]: In the 'zero slope' limit the superstring leads to a chiral 10-dimensional supergravity theory, and anomaly cancellation fixes the gauge group almost uniquely. The Kaluza–Klein scenario thus eventually obtained a solid basis, but this time including an (almost) unique additional gauge group $E_8 \times E_8$ (or SO_{32}). As it turned out, however, the vacuum structure is not so unique after compactification or when string theories are constructed directly in 4 dimensions. There remain many open problems concerning the quantum mechanics that (hopefully) selects a ground state resembling the observable universe (which includes a small cosmological term *after* SUSY breaking). Moreover, it is still not at all clear what string theory really is.

[5] 'Color strings' are, however, still in use for describing quark interaction at long distances; 'cosmic strings' could form from topological defects in spontaneous symmetry breaking. In both cases, the Nambu–Goto action is only an approximation. We will only be interested in 'fundamental' strings.

[6] It turned out that the GSO projection is not only possible but is mandatory in order to avoid global anomalies at higher genus (this requirement is called 'modular invariance'). Thus supersymmetry, and in particular the presence of fermions, presumably is an unavoidable consequence of string unification.

Chapter 2

The bosonic string

The Nambu–Goto action for the bosonic string is given by the area of the world sheet that is embedded in some D-dimensional space. If the target space is itself a general manifold with a metric $G_{\mu\nu}(X)$ depending on local coordinates X^μ then the resulting theory is called a (non-linear) σ-model. So we start with the action[1]

$$S_N[X] = -T\int d^2\sigma\sqrt{-\det G^*} \qquad \text{with} \;\; G^*_{mn} := (X^*G)_{mn} = \frac{\partial X^\mu}{\partial\sigma^m}\frac{\partial X^\nu}{\partial\sigma^n}G_{\mu\nu}(X), \tag{2.1}$$

where σ^0 and σ^1 are local coordinates of the world sheet and the embedding is described by D coordinate functions $X^\mu(\sigma)$. The *induced metric* $G^*_{mn}(\sigma)$ on the world sheet is the *pull back* X^*G of the target space metric $G_{\mu\nu}(X)$ to the parameter space of the embedded surface. The *string tension* T is a constant with the dimension of an inverse length squared and the sign of the action is chosen such that the kinetic energy will be positive for the space-like coordinates $X^1,\ldots,X^{D-1}$ of the target space (see below).

2.1 The Polyakov action

From the σ-model point of view another natural action for the string is

$$S_P[X,g] = -\frac{T}{2}\int d^2\sigma\sqrt{-g}\; g^{mn}\frac{\partial X^\mu}{\partial\sigma^m}\frac{\partial X^\nu}{\partial\sigma^n}G_{\mu\nu}. \tag{2.2}$$

Although this action was already used in refs. [de76, br76] as the bosonic part of a supersymmetric action for the spinning string, it usually goes under the name *Polyakov*, who emphasized the role of the 2-dimensional geometry on the world sheet and showed how to quantize the string

[1] The area element spanned by the vectors $\partial_1 X$ and $\partial_2 X$, which are tangential to target space, is given by $|\partial_1 X|\,|\partial_2 X|\sin(\partial_1 X,\partial_2 X) = |\partial_1 X|\,|\partial_2 X|\sqrt{1-\left(\frac{\partial_1 X\cdot\partial_2 X}{|\partial_1 X|\,|\partial_2 X|}\right)^2} = \sqrt{\det\left(\frac{\partial X}{\partial\sigma^m}\cdot\frac{\partial X}{\partial\sigma^n}\right)}$, where $\sin(\partial_1 X,\partial_2 X)$ denotes the sinus of the angle between the tangent vectors.

in arbitrary dimensions [po81]. The two actions (2.1) and (2.2) are classically equivalent, as they lead to the same equations of motion for the string world sheet. To see this we calculate the variation of S_P with respect to the metric, which by definition[2] is proportional to the energy–momentum tensor T_{mn}. Using $\delta(\ln\det M)=\delta(\mathrm{tr}\ln M)=\mathrm{tr}(M^{-1}\delta M)$ we obtain

$$T_{mn}:=\frac{2}{T}\frac{1}{\sqrt{-g}}\,\frac{\delta S_P}{\delta g^{mn}}=\frac{1}{2}g_{mn}g^{kl}G^*_{kl}-G^*_{mn}=0. \tag{2.3}$$

The equation of motion $T_{mn}=0$ implies that the world sheet metric must be proportional to the induced metric, i.e. $g_{mn}=\rho G^*_{mn}$, where the factor $\rho=2/(g^{kl}G^*_{kl})$ drops out of all equations and remains arbitrary. Since the Polyakov action does not depend on derivatives of the metric, $\frac{\delta S_P}{\delta g^{mn}}=0$ is algebraic in g_{mn} and we may insert it back into the action without changing the equations of motion for the 'matter fields' X^μ. Taking determinants, we thus observe classical equivalence. For quantization, however, S_P is more convenient because the world sheet scalars X^μ now have their usual kinetic terms rather than appearing in the square root of a determinant.

Now we compute the total variation of the action to obtain the equations of motion[3] for minimal area surfaces (since $G_{\mu\nu}$ depends on $X^\alpha(\sigma^m)$ we have $\partial_m G_{\mu\nu}=\partial_m X^\alpha\partial_a G_{\mu\nu}$):

$$-\tfrac{2}{T}\,\delta S_P=\int d^2\sigma\,\sqrt{-g}\,\Big(\delta X^\alpha\partial_\alpha G_{\mu\nu}D_nX^\mu D^nX^\nu+2D_n(\delta X^\alpha)D^nX^\rho\,G_{\alpha\rho}-\delta g^{mn}T_{mn}\Big) \tag{2.4}$$

$$=\int d^2\sigma\,\sqrt{-g}\,\Big(\delta X^\alpha\Big(\partial_\alpha G_{\mu\nu}D_nX^\mu D^nX^\nu-2D^2X^\rho G_{\alpha\rho}-2D^nX^\rho D_nX^\nu\partial_\nu G_{\alpha\rho}\big)-\delta g^{mn}T_{mn}\Big)$$

Here we ignored surface terms which have to be taken into account for open strings (see below). The last term $\partial_\nu G_{\alpha\rho}$ is symmetrized in ν and ρ, hence all derivatives of the target space metric combine to give the Christoffel symbol $\hat\Gamma_{\mu\nu\alpha}=\frac{1}{2}(\partial_\mu G_{\alpha\nu}+\partial_\nu G_{\alpha\mu}-\partial_\alpha G_{\mu\nu})$ of the target space metric. Contracting $\delta S_P/\delta X^\alpha$ with $G^{\lambda\alpha}$ we thus arrive at the equations of motion

$$\Delta X^\lambda+g^{mn}\partial_m X^\mu\partial_n X^\nu\hat\Gamma_{\mu\nu}{}^\lambda=0, \tag{2.5}$$

$\Delta:=D^2=g^{mn}D_mD_n=g^{mn}(\partial_m\partial_n-\hat\Gamma_{mn}{}^l\partial_l)$ is the Laplace–Beltrami operator for scalars on the world sheet. (In a non-covariant evaluation of the variational derivative the Christoffel symbol $\hat\Gamma_{mn}{}^l$ of the world sheet metric comes from the term $\partial_m(\sqrt{g}\,g^{mn})=-\sqrt{g}\,g^{kl}\hat\Gamma_{kl}{}^n$.) Note that we recover the geodesic equation if we let the string collapse to a point, so that all derivatives with respect to the space coordinate σ^1 are zero, and use an affine parametrization $g_{00}=1$ of the resulting world line.

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[2] In string theory it is common to deviate from the usual normalization $T_{mn}=\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{mn}}$, which is consistent with the Noether formula $\hat T_l{}^m=\partial_l\phi^i\frac{\partial\mathcal{L}}{\partial\partial_m\phi^i}-\delta_l^m\mathcal{L}$ for the canonical energy–momentum tensor in Minkowski space. $\hat T^{nm}=\eta^{nl}\hat T_l{}^m$ is in general neither symmetric nor gauge invariant and differs from the flat space limit of T^{mn} by the Belinfante improvement term, which is a divergence of an antisymmetric tensor, plus terms that are proportional to the equations of motion.

[3] The variational derivatives of an action $S=\int\mathcal{L}(\phi,\partial_m\phi)$ are defined by $\frac{\delta S}{\delta\phi^i}:=\frac{\partial\mathcal{L}}{\partial\phi^i}-\partial_m\frac{\partial\mathcal{L}}{\partial\partial_m\phi^i}$. In curved space it is, however, often more efficient to compute the variation directly with covariant partial integration in scalar densities using $\int\partial_{m_1}(\sqrt{g}\,A_{m_2\ldots m_I}B^{m_1\ldots m_I})=\int\sqrt{g}\,D_{m_1}(A_{m_2\ldots m_I}B^{m_1\ldots m_I})=\int\sqrt{g}\,(D_{m_1}A_{m_2\ldots m_I}B^{m_1\ldots m_I}+A_{m_2\ldots m_I}D_{m_1}B^{m_1\ldots m_I})$.

2.2 Local symmetries and gauge fixing

Before we try to solve the equations of motion we should have a look at the symmetries of the Polyakov action $\mathcal{L}_P$. By construction the Nambu–Goto action is coordinate invariant in the target space as well as on the world sheet. This carries over to $\mathcal{L}_P$, but for that action we have, in addition, the Weyl invariance $g_{mn}(\sigma) \to e^{2\Lambda(\sigma)}\, g_{mn}(\sigma)$ on the world sheet. Together with two coordinate functions $\tilde{\sigma}^m(\sigma)$ this gives a total of 3 functions of σ^m that we are free to choose.

The number of gauge degrees of freedom thus coincides with the degrees of freedom in the world sheet metric. This suggests that we should be able to use a flat metric $g_{mn} = \eta_{mn}$ on the world sheet, which indeed is true locally. To see this note that in two dimensions for any two linearly independent vector fields there exists a coordinate system whose coordinate lines coincide with the integral curves of the vector fields. Having a metric with Lorentzian signature, there are two natural vector fields defined by the two independent null vectors at each point. In a corresponding coordinate system with coordinates σ^+ and σ^- the metric has only off-diagonal entries. With $\partial_\pm := \partial/\partial\sigma^\pm$ we thus have

$$g_{+-} = g(\partial_+, \partial_-) = \tfrac{1}{2}e^{\varphi}, \qquad g_{++} = g_{--} = 0. \tag{2.6}$$

$\sigma^\pm$ are called light-cone coordinates. They are unique up to reparametrizations $\sigma^\pm \to f_\pm(\sigma^\pm)$ and we choose them in such a way that $\tau = \sigma^0 := (\sigma^+ + \sigma^-)/2$ is time-like and increasing with the target-space time X^0, whereas $\sigma = \sigma^1 := (\sigma^+ - \sigma^-)/2$ is space-like. $g_{+-} > 0$ is required by $g_{00} = g_{++} + 2g_{+-} + g_{--} > 0$ and $g_{11} = g_{++} - 2g_{+-} + g_{--} < 0$. These equations, as well as $g_{01} = g_{++} - g_{--}$, follow from $\partial_\tau = \partial_+ + \partial_-$ and $\partial_\sigma = \partial_+ - \partial_-$. We also find

$$\sigma^\pm = \tau \pm \sigma, \qquad g^{+-} = 2e^{-\varphi} = 1/g_{+-}, \qquad g_{+-} = \tfrac{1}{4}(g_{00} - g_{11}), \tag{2.7}$$

$$\partial_\pm = \tfrac{1}{2}(\partial_\tau \pm \partial_\sigma), \qquad g_{mn} = e^{\varphi}\eta_{mn}, \qquad g_{\pm\pm} = \tfrac{1}{4}(g_{00} \pm 2g_{01} + g_{11}). \tag{2.8}$$

Now we can perform a Weyl rescaling to get a flat world sheet metric $g_{mn} = \eta_{mn}$.

In light-cone coordinates we obtain very simple expressions for the Christoffel symbol, whose only non-vanishing components are

$$\hat{\Gamma}_{++}{}^{+} = \partial_+ \ln\sqrt{-g} = \partial_+\varphi, \qquad \hat{\Gamma}_{--}{}^{-} = \partial_- \ln\sqrt{-g} = \partial_-\varphi, \tag{2.9}$$

since $\hat{\Gamma}_{m++} = \hat{\Gamma}_{m--} = 0$. For the energy–momentum tensor (2.3) we find

$$T_{++} = -\partial_+ X \cdot \partial_+ X, \qquad T_{+-} = 0, \qquad T_{--} = -\partial_- X \cdot \partial_- X. \tag{2.10}$$

There also is a simple geometrical interpretation of the minimal area equation: Observing that $\partial_+ X^\mu$ and $\partial_- X^\mu$ are the light-like tangent vectors defined by the coordinate lines σ_+ and σ_- it is easy to see that (2.5) is nothing but $D_{\partial_- X}\partial_+ X = 0$, i.e. the covariant derivative (with respect

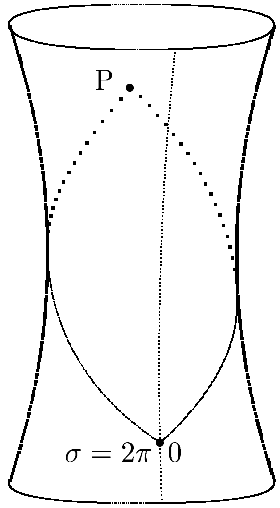

Fig. 4: Closed string with $0 < \sigma < 2\pi$

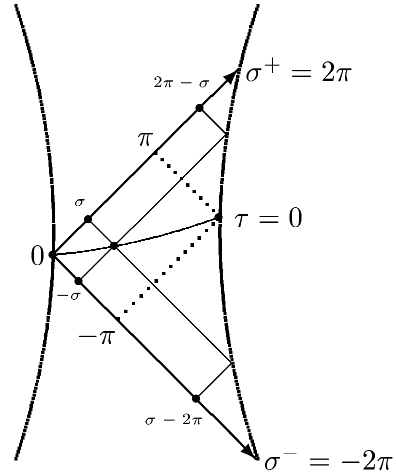

Fig. 5: Open string with $0 < \sigma < \pi$

to the Levi–Civita connection in target space) of $\partial_+ X$ along $\partial_- X$ has to vanish. For a flat target space $G_{\mu\nu} = \eta_{\mu\nu}$ this reduces to the wave equation $\Box X^\mu = \partial_+ \partial_- X^\mu = 0$, whose general solution is a superposition of a left-moving mode $X_L^\mu(\sigma^+)$ and a right-moving mode $X_R^\mu(\sigma^-)$.

We now turn to global properties of our choice of parametrization. The basic assumption for our local construction was that the metric has Minkowski signature. This cannot be true globally for interacting strings, as can be seen for a 'pant' representing a smooth joining of two closed strings: For such a world sheet there is always some region where the induced metric is Euclidean. It is therefore convenient to restrict our attention to free strings with 'generic' world sheets and to postpone the study of the interacting case and the rigorous treatment of global questions till after a Wick rotation to Euclidean space. In particular, we exclude world sheets with closed time-like curves and degenerations of the light cone.

The choice of light cone coodinates still allows for reparametrizations of the coordinate lines $\sigma^\pm \to f^\pm(\sigma^\pm)$. This freedom can now be used to choose a parametrization that is 2π-periodic in σ for closed strings:

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi) \tag{2.11}$$

(the case of open strings will be discussed in section 2.3). In order to see that (2.11) is consistent with the conformal gauge $g_{mn} \propto \eta_{mn}$ we choose an arbitrary point on the closed string surface as the coordinate origin $\sigma = \tau = 0$ (see Fig. 4). Then we go along the two light-like curves in positive time direction till we arrive at the first intersection point P and choose some parametrization of these two pieces of coordinate lines in such a way that the coordinate labels are smooth and monotonic and reach 2π at P. We can now assign coordinates $\sigma^\pm$ modulo 2π to any point on the surface by going along the two light cones till we meet one of the sections of coordinate lines where the coordinates have been chosen. In order to fix the coordinates completely we cut the surface along a time-like curve through the origin and demand that the coordinate functions are continuous on the resulting strip. In order to see what happens at the cut we go from the origin to the intersection point along the σ^+ coordinate line in positive direction till we reach the point P with $\sigma^+ = 2\pi$ and $\sigma^- = 0$. Then we continue along the other coordinate line till we return to the origin, with σ^+ constant and σ^- decreasing by 2π till we arrive at the origin. We thus observe that the coordinate σ jumps by 2π and that τ is continuous if we cross the cut at the origin. Because of continuity of the coordinates away from the cut the same has to happen everywhere along the cut.

If we restrict ourselves to parametrizations satisfying (2.11) then we are only free to choose a parametrization of the coordinate lines $\sigma^\pm$ in the intervals $0 < \sigma^\pm < 2\pi$. Taking into account the additional freedom of choosing the origin and imposing smoothness of the coordinate transformation at the origin we end up with a residual reparametrization freedom $\sigma^\pm \to \sigma^\pm - \xi^\pm$ with smooth 2π-periodic functions $\xi^+(\sigma^+)$ and $\xi^-(\sigma^-)$, where we also require $|\partial_\pm \xi^\pm| < 1$ in order that the new coordinates be monotonic.

The restriction that the parametrization of the surface should be 2π-periodic in σ with the metric g_{mn} being proportional to η_{mn} is called *conformal gauge*. A diffeomorphism $\tilde\sigma^m = \sigma^m - \xi^m(\sigma)$ that changes the metric only by a Weyl rescaling $\tilde g_{mn}(\tilde\sigma) = e^{2\Lambda(\sigma)} g_{mn}(\sigma)$ is called a *conformal transformation* (in other words, this is a reparametrization that preserves angles between coordinate lines; of course we may also think about such a transformation in an active way as giving us a new surface parametrized by the old coordinates). Such transformations respect the conformal gauge. Considering infinitesimal transformations (i.e. keeping only terms that are linear in the small quantities ξ^m and Λ) this leads to the *conformal Killing equation*

$$\delta g_{mn} = \mathcal{L}_\xi g_{mn} - 2\Lambda g_{mn} = D_m \xi_n + D_n \xi_m - 2\Lambda g_{mn} = 0. \tag{2.12}$$

This means that variation of the metric under infinitesimal coordinate transformations, which is given by the Lie derivative $\mathcal{L}_\xi g_{mn} = D_m\xi_n + D_n\xi_m$ with respect to the vector field ξ^m, can be compensated by a Weyl transformation. Taking the trace we find $D_n\xi^n = \Lambda g^{mn} g_{mn} = \Lambda\, d$, so that the Weyl factor becomes proportional to the covariant divergence of ξ. In $d = 2$ dimensions this yields $D_m\xi_n + D_n\xi_m - g_{mn} D_l \xi^l = 0$. Using light cone coordinates this equation is an identity for $(m, n) = (\pm, \mp)$ and we recover the conditions $D_\pm \xi_\pm = g_{+-} D_\pm \xi^\mp = 0 \;\Leftrightarrow\; \partial_\pm \xi^\mp = 0$.

2.3 Open strings

For open strings the Euler-Lagrange equations of motion still have to be supplemented by boundary conditions that come from the surface terms of a general variation of the action. But first we construct, in analogy to our discussion of closed strings, a conformally flat coordinate system whose space coordinate σ ranges from 0 to π for all τ. To this end we choose a point at the left boundary[4] as the origin and choose coordinate labels $0 < \sigma^+ < 2\pi$ along the future light cone, as shown in Fig. 5. Imposing the condition that the left and the right boundary of the string is parametrized by $\sigma = 0$ and $\sigma = \pi$, respectively, we can assign coordinates to all points on the string surface by following the light rays till we intersect the original piece of σ^+-coordinate line. The only difference to the case of closed strings is that this time the coordinate lines are 'reflected' at the boundary. On the σ^- coordinate line through the origin, for example, we find coordinate labels between 0 and -2π ($\sigma^\pm$ is constant along the $\sigma^\mp$ coordinate lines).

From this construction it follows that the functions $f^+(\sigma^+)$ and $f^-(\sigma^-)$ that correspond to the residual gauge invariance in conformal gauge must be identical $f^+ \equiv f^-$ and 2π periodic to be consistent with a σ coordinate that runs from 0 to π. The freedom of parametrizing the σ^+ coordinate line (and thereby also the σ^- line) can also be interpreted in a different way: As a consequence of the choice of $\sigma^\pm$ labels the line $\tau = (\sigma^+ + \sigma^-)/2 = 0$ and the σ coordinate labels on that line are fixed. In turn, we can first choose the line of vanishing time $\tau = 0$ and assign σ coordinate labels between 0 and π on that line. Then the $\sigma^\pm$ coordinate labels can be constructed as shown in Fig. 5. Hence the choice of the line of equal time $\tau = \tau_0$ corresponds to the even part of 2π-periodic infinitesimal reparametrizations $\delta\sigma^\pm = f(\sigma^\pm)$, and the freedom of assigning the σ coordinate labels corresponds to f_{odd} in the unique decomposition $f = f_{even} + f_{odd}$.

Now we turn to the derivation of the boundary conditions at the ends of the string. We require that the action should be stationary if the variation vanishes at the initial and final times, but is arbitrary at the string ends. To avoid terms coming from a variation of the integration domain we assume a parametrization with $0 < \sigma < \pi$. We thus pick up a surface term

$$\int d^2\sigma \partial_m \left(\delta X^\mu \frac{\partial S}{\partial \partial_m X^\mu} \right) = -T \int_{t_0}^{t_1} d\tau \, \left(\delta X^\mu \sqrt{-g}\, G_{\mu\nu}\, g^{1n} \partial_n X^\nu \right) \Big|_{\sigma=0}^{\sigma=\pi}. \tag{2.13}$$

If we require this expression to vanish for arbitrary variations δX^μ we conclude that for $\sigma = 0$ and for $\sigma = \pi$ we must have

$$\sqrt{-g}\,(g^{10}\dot{X} + g^{11}X') = (g_{01}\dot{X} - g_{00}X')/\sqrt{-g}\; = 0. \tag{2.14}$$

This implies that the induced metric G^*_{mn} becomes singular at the boundary, which can be seen as follows: First assume that $\dot{X}$ and X' are linearly independent, implying that $g_{01}/\sqrt{-g} = g_{00}/\sqrt{-g} = 0$. The equations of motion imply that g_{mn} is proportional to the induced metric $G^*_{mn} = \rho g_{mn}$ everywhere on the world sheet, with ρ dropping out in the ratio $g_{mn}/\sqrt{-g}$. This cannot happen if G^*_{mn} has a non-singular limit at the boundary. If, on the other hand, $\dot{X}$ and

[4] 'left' refers to decreasing space-like coordinate σ^1 for an oriented parametrization with X^0 increasing with a time-like coordinate σ^0.

X' become proportional at the boundary, then the matrix $G^*_{mn} = \partial_m X \cdot \partial_n X$ also is singular. The discussion of boundary conditions is therefore very delicate in general and we better first choose a convenient gauge.

In conformal gauge the induced metric G^*_{mn}, with entries $\dot{X}^2$, $\dot{X} \cdot X'$ and $(X')^2$, becomes proportional to the flat metric, i.e. $G^*_{mn} = \rho\,\eta_{mn}$, and ρ has to vanish at the boundary. Inserting this back into eq. (2.14) we find the boundary conditions $X' = 0$ and $\dot{X}^2 = 0$. The second condition has the geometrical interpretation that the string ends move with the speed of light, and therefore is independent of the gauge. The D conditions $\partial_\sigma X^\mu = 0$, on the other hand, are valid only in the conformal gauge, as can be seen by choosing a gauge for which the coordinate lines are not orthogonal.[5] Sticking to the conformal gauge, we have *von Neumann* boundary conditions. We therefore can continue the coordinate functions $X^\mu(\tau, \sigma)$ beyond $0 < \sigma < \pi$ to get even and 2π-periodic functions of σ. Hence all open string solutions can be obtained in the conformal gauge as special cases of closed string solutions.

Von Neumann boundary conditions (in conformal gauge) are the only Lorentz-invariant possibility to make surface terms vanish. If we relax that condition, however, it is also possible to make (2.14) vanish by choosing Dirichlet boundary conditions $X^\mu = const.$ for p of the space-like string coordinates (in a flat target space). The string ends are then constrained to move on a p-dimensional submanifold, a so-called D-brane (a p-brane is an extended object of space-time dimension $p + 1$, so that a 2-brane is a membrane and a 1-brane is a string; here, however, the 'D' stands for 'Dirichlet', indicating that open strings have to end on that brane without specifying its dimension). The consistency and importance of these boundary conditions was discovered in the context of T-duality [da89, ho89] (see below) and the presense and dynamics of the associated (solitonic) extended objects, i.e. p-branes acting as D-branes, plays an important role in recent results on non-perturbative string dualities [po$_1$95, po$_1$96].

2.4 Target space symmetries and conservation laws

By construction, the Polyakov action is invariant under arbitrary coordinate transformations of the world sheet and of the target space. The local world sheet invariances imply gauge symmetries of the action, as we discussed above. Target space coordinate invariance, on the other hand, in general does not imply any symmetry of the σ model, since $\mathcal{L}_P$ is only invariant if we also transform $G_{\mu\nu}$. Since the functions $G_{\mu\nu}(X)$ can be interpreted as (an infinite number of) coupling constants, target space coordinate transformations relate *different* σ models by

[5] Consider, for example, the solution $X^\mu = (\tau, \cos\sigma\cos\tau, \cos\sigma\sin\tau, 0, \ldots)$ to the equations of motion in conformal-gauge, which satisfies all boundary conditions. Changing the parametrization by $\tau = \overline{\tau} + a\sigma$ we find $\dot{X}^2 = \sin^2\sigma$, $X'^2 = (a^2 - 1)\sin^2\sigma$, and $\dot{X}\,X' = a\sin^2\sigma$. Then $X' = (a, -a\cos\sigma\sin\overline{\tau} - \sin\sigma\cos\overline{\tau}, a\cos\sigma\cos\overline{\tau} - \sin\sigma\sin\overline{\tau}, \ldots)$, which shows that X' does not have to vanish at the boundary in a general gauge.

a reparametrization of the dynamical fields X^μ. We do, however, have a symmetry of the σ model if the new functions $G'_{\mu\nu}(X')$ turn out to be identical to the old metric $G_{\mu\nu}(X)$. Then the target manifold has a (geometrical) symmetry, which corresponds to a global symmetry of the σ model, because the transformation $X \to X'$ of the dynamical fields is independent of σ.

Continuous target space symmetries are equivalent to the global existence of Killing vector fields $\Xi^\mu(X)$ with $\mathcal{L}_\Xi G_{\mu\nu} = D_\mu \Xi_\nu + D_\nu \Xi_\mu = 0$. According to the Noether theorem[6] they imply the existence of conserved quantities. In the case of a flat target space, for example, we have $G_{\mu\nu} = \eta_{\mu\nu}$ and the general solution to the Killing equation is

$$\Xi^\mu = A^\mu + X^\nu \Omega_\nu{}^\mu \tag{2.15}$$

with $\Omega_{\mu\nu}$ antisymmetric. Invariance under the D independent translations $\delta_\mu X^\rho = -\delta_\mu^\rho$ implies conservation of the target space energy–momentum currents P_μ^m with the corresponding conserved charges $P_\mu = \int d\sigma P_\mu^0$ (for convenience we use the conformal gauge):

$$P_\mu^m = \delta_\mu^\rho \frac{\partial \mathcal{L}}{\partial \partial_m X^\rho} = -T\eta^{mn}\partial_n X_\mu, \qquad P_\mu = -T\int d\sigma\, \dot{X}_\mu. \tag{2.16}$$

Note that the object P_μ^m is different from the canonical (flat) *world sheet* energy–momentum tensor $\hat{T}_l{}^m = K_l{}^m - \delta_l X^\mu \frac{\partial\mathcal{L}}{\partial\partial_m X^\mu} = -T(\partial_l X^\mu \partial^m X_\mu - \frac{1}{2}\delta_l^m \delta_k^j \partial_j X^\mu \partial^k X_\mu)$, where $\delta_l \mathcal{L} = \partial_m K_l{}^m = -\partial_l \mathcal{L}$ is the infinitesimal change of the Lagrangian under translations $\delta_l \phi^i = -\partial_l \phi^i$, so that $K_l{}^m = -\delta_l{}^m \mathcal{L}$. In particular, $H = \int d\sigma\, \hat{T}_0{}^0 = \int d\sigma(\dot{X}^\mu \Pi_\mu - \mathcal{L})$, with the canonical momenta $\Pi_\mu = \partial\mathcal{L}/\partial\dot{X}^\mu = -T\dot{X}_\mu$, is the Hamiltonian of our 2-dimensional field theory, which generates time translations (up to a factor T, which is due to our convention in eq. (2.3), $\hat{T}_l{}^m$ is the flat limit of $T_l{}^m$).

Infinitesimal Lorentz transformations $\delta_{\mu\nu} X^\rho = \delta_\mu^\rho X_\nu - \delta_\nu^\rho X_\mu$ yield the angular momenta

$$J_{\mu\nu}^m = (X_\mu \delta_\nu^\rho - X_\nu \delta_\mu^\rho)\frac{\partial\mathcal{L}}{\partial\partial_m X^\rho} = -T\left(X_\mu \partial^m X_\nu - X_\nu \partial^m X_\mu\right), \quad J_{\mu\nu} = -T\int d\sigma \left(X_\mu \dot{X}_\nu - X_\nu \dot{X}_\mu\right). \tag{2.17}$$

The Poisson brackets of these charges represent the Poincaré algebra (we include a factor i since $i\{A,B\}_{PB}$ will be replaced by the commutator $[A,B]$ upon quantization):

$$\{P_\alpha, P_\beta\}_{PB} = 0, \qquad i\{J_{\mu\nu}, P_\alpha\}_{PB} = i\eta_{\mu\alpha}P_\nu - i\eta_{\nu\alpha}P_\mu, \tag{2.18}$$

$$i\{J_{\mu\nu}, J_{\alpha\beta}\}_{PB} = i\eta_{\mu\alpha}J_{\nu\beta} - i\eta_{\nu\alpha}J_{\mu\beta} - i\eta_{\mu\beta}J_{\nu\alpha} + i\eta_{\nu\beta}J_{\mu\alpha}. \tag{2.19}$$

The brackets among coordinates and momenta are $\{\Pi_\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{PB} = -\delta(\sigma-\sigma')\delta_\mu{}^\nu$.

[6] The first Noether theorem states that continuous symmetries are in one-to-one correspondence with conserved charges: In a field theory with an action $S = \int d^4x \mathcal{L}(\phi^i, \partial\phi^i)$ that is invariant under the infinitesimal transformations $\delta_I \phi^i = R_I{}^i(\phi, \partial\phi)$, i.e. with $\mathcal{L}$ transforming into total derivatives $\delta_I \mathcal{L} = \partial_m K_I{}^m$, the explicit formula for the corresponding Noether currents is $J_I{}^m := K_I{}^m - \delta_I \phi \frac{\partial\mathcal{L}}{\partial\partial_m \phi}$. Since $\delta_I \mathcal{L} = \partial K_I = \delta_I \phi^i \left(\frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_m \frac{\partial\mathcal{L}}{\partial(\partial_m\phi^i)}\right) + \partial_m\left(\delta_I \phi^i \frac{\partial\mathcal{L}}{\partial(\partial_m\phi^i)}\right)$ the equations of motion imply that the divergence $\partial_m J_I{}^m$ vanishes, i.e. the currents J_I are *conserved*, so that the charges $Q_I = \int d^3x\, J_I{}^0$ are time independent on shell up to surface terms $\dot{Q}_I = \int d^3x\, \vec{\partial}\vec{J}_I$.

2.5 Classical solutions and light cone gauge

Now we want to solve the equations of motion, so we restrict our discussion to the case of a flat target space and use the conformal gauge with the appropriate boundary conditions. Then the coordinate functions fulfill $\partial_+\partial_- X^\mu = 0$ with $\sigma^\pm = \tau \pm \sigma$, hence $X^\mu(\tau,\sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$ with

$$\partial_+ X^\mu = \partial_+ X_L^\mu = \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{\sqrt{4\pi T}} e^{-in\sigma^+}, \qquad \partial_- X^\mu = \partial_- X_R^\mu = \sum_{n=-\infty}^{\infty} \frac{\tilde\alpha_n^\mu}{\sqrt{4\pi T}} e^{-in\sigma^-}. \tag{2.20}$$

Integrating these equations we obtain an integration constant, which we choose to be equal $x_L^\mu = x_R^\mu = x^\mu$ for X_L and X_R. The boundary conditions imply that the zero modes α_0^μ and $\tilde\alpha_0^\mu$ must be equal, because there can be no linear σ dependence for a periodic function, and we set $\alpha_0^\mu = \tilde\alpha_0^\mu = p^\mu/\sqrt{4\pi T}$. Hence

$$X_L^\mu(\tau+\sigma) = \frac{1}{2} x_L^\mu + \frac{1}{4\pi T} p^\mu\, \sigma^+ + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\sigma^+}, \tag{2.21}$$

$$X_R^\mu(\tau-\sigma) = \frac{1}{2} x_R^\mu + \frac{1}{4\pi T} p^\mu\, \sigma^- + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{1}{n} \tilde\alpha_n^\mu e^{-in\sigma^-} \tag{2.22}$$

is the most general solution.[7] Reality of the coordinate functions X^μ implies $\alpha_n^* = \alpha_{-n}$. From the closed string solutions we obtain all solutions for open strings by restricting to even functions. This means that for open strings the left-moving and the right-moving modes must be equal $\tilde\alpha_n = \alpha_n$ and we have much less freedom.

We must be careful to remember that so far we only fulfilled the equations of motion $\delta S/\delta X^\mu = 0$. We still have to set the energy–momentum tensor to zero, i.e. we must impose $T_{++} = -(\partial_+ X)^2 = 0$ and $T_{--} = -(\partial_- X)^2 = 0$: For the left-moving part this means that

$$T_{++} = \frac{1}{2\pi T} \sum_{n=-\infty}^{\infty} L_n e^{-in\sigma^+} = 0, \qquad T_{--} = \frac{1}{2\pi T} \sum_{n=-\infty}^{\infty} \tilde L_n e^{-in\sigma^-} = 0. \tag{2.23}$$

The Virasoro generators $L_n := T \int_0^{2\pi} d\sigma^+ T_{++} e^{in\sigma^+}$ and $\tilde L_n := T \int_0^{2\pi} d\sigma^- T_{--} e^{in\sigma^-}$ are given by

$$L_n = -\frac{1}{2} \sum_{m=-\infty}^{\infty} \eta_{\mu\nu} \alpha_m^\mu \alpha_{n-m}^\nu, \qquad \tilde L_n = -\frac{1}{2} \sum_{m=-\infty}^{\infty} \eta_{\mu\nu} \tilde\alpha_m^\mu \tilde\alpha_{n-m}^\nu. \tag{2.24}$$

[7] The situation is different for compactified dimensions: If X^μ lives on a circle of radius R then we must allow $X^\mu(\sigma+2\pi) = 2R\pi n + X^\mu(\sigma)$ since the string may wind n times around the loop. Then $\frac{1}{2T}(p_L^\mu - p_R^\mu) = n\,2\pi R$ for some $n \in \mathbb{Z}$. The total momentum $P^\mu = p_L^\mu + p_R^\mu$, on the other hand, will be quantized in units of $1/R$ in the quantum theory because $\exp(2\pi i\, RP^\mu)$ generates a translation by $2\pi R$ and thus has to be the identity operator. This is our first indication of the large/small radius duality $R \to 1/(4\pi T\, R)$. Upon quantization $p_L - p_R$ becomes the winding number operator. Choosing arbitrary integration constants x_L^μ and x_R^μ, the *collective coordinates* are $x^\mu = \frac{1}{2}(x_L^\mu + x_R^\mu)$ and we may use the combination $x_L^\mu - x_R^\mu$, which does not contribute to X^μ, as the conjugate variable for the winding number. In this way we decompose the operator algebra into a left-moving and a right-moving part. For uncompactified dimensions, on the other hand, left-movers and right-movers are always coupled through the momentum zero modes $p_L^\mu = p_R^\mu$.

The constraints $L_0 = 0 = \tilde{L}_0$, which generate global translations on the world sheet (see below), play a special role. Inserting the definition $p^\mu = \sqrt{4\pi T}\alpha_0^\mu$ for the zero mode, they read

$$p^2 = -8\pi T \sum_{n>0} \alpha_n^* \cdot \alpha_n = -8\pi T \sum_{n>0} \tilde{\alpha}_n^* \cdot \tilde{\alpha}_n. \tag{2.25}$$

Vanishing of $H = (L_0 + \tilde{L}_0)/2$, the 2-dimensional Hamiltonian, tells us the mass $m^2 = P^2$ of a string in terms of the oscillators α_n with $n \neq 0$ (recall that $P^2 = p^2/4$ in case of open strings). This constraint is called the mass shell condition; the generator $L_0 - \tilde{L}_0$ of translations in the space direction equates the masses of left and right movers.

The Virasoro constraints $L_n = 0$ are infinitely many quadratic equations and hard to solve in general. It is therefore time to remember that we still have some gauge freedom left, which we may use to simplify these equations. Note that the periodic reparametrizations of the light-cone coordinates, which are still allowed, lead to the freedom $\tau \to \frac{1}{2}(f^+(\sigma^+) + f^-(\sigma^-))$, which just corresponds to a solution of the wave equation for the coordinate functions. We may therefore choose τ proportional to $c_\mu X^\mu$ for some fixed time-like or light-like vector c_μ (a space-like c_μ would lead to a space-like time direction on the world sheet). Because of the identity

$$V^\pm = V^0 \pm V^{D-1} \qquad \Rightarrow \qquad V_\mu W^\mu = \frac{1}{2}(V^+W^- + V^-W^+) - \sum_{i=1}^{D-2} V^i W^i \tag{2.26}$$

a light-like choice $c_\mu = (1, 0 \ldots, 0, 1)$ is particularly useful. In the resulting **light cone gauge** we impose $X^+ = 2\pi T\; p^+\tau$, which implies that all oscillator coefficients $\alpha_n^+ = \tilde{\alpha}_n^+ = 0$ vanish for $n \neq 0$. Now the Virasoro constraints obtain linear terms and can be solved explicitly for

$$\alpha_n^- = \frac{\sqrt{4\pi T}}{p^+} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} \alpha_m^i \alpha_{n-m}^i, \qquad \tilde{\alpha}_n^- = \frac{\sqrt{4\pi T}}{p^+} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} \tilde{\alpha}_m^i \tilde{\alpha}_{n-m}^i \tag{2.27}$$

(recall that $p^+ = \alpha_0^+\sqrt{4\pi T}$). This gauge, however, abandons manifest Lorentz invariance in target space and it turns out that the quantum theory violates the Lorentz algebra (2.19) if $D \neq 26$ [go73]; historically this was the first derivation of the critical dimension of the bosonic string. Note that the light cone gauge assumes $p^+ \neq 0$, which can always be achieved by a Lorentz transformation unless $p^\mu \equiv 0$. This is o.k. for a single free string, but in case of interactions intermediate strings may have arbitrary momenta and we should expect sublte technical problems in perturbation theory [gr88].

In order to find a simple solution to the equations of motion (including the Virasoro constraints) we assume that only one frequency is excited (i.e. only one oscillator α_n and its complex conjugate α_{-n}, as well as the zero mode p^μ, are non-zero). Then the only relevant left-moving constraints are

$$-L_0 = \tfrac{1}{2}|\alpha_0|^2 + \alpha_{-n} \cdot \alpha_n = 0, \qquad -L_n = \alpha_0 \cdot \alpha_n = 0, \qquad -L_{2n} = \tfrac{1}{2}\alpha_n \cdot \alpha_n = 0. \tag{2.28}$$

The first constraint is the mass shell condition, $L_n = 0$ implies transversality with respect to the momentum p^μ, and the third condition implies a light-like polarization vector α_n; by complex conjugation $L_{-n} = 0$ and $L_{-2n} = 0$ are redundant. To simplify things further we keep the string oscillation in the $X^1 - X^2$ plane, and we go to the rest frame and set $x^\mu = \vec{p} = 0$. We now use the light cone gauge[8] and choose $\alpha_n^\mu = \rho\frac{n\sqrt{\pi T}}{2}(0,1,i,0\ldots)$ and $\tilde\alpha_n^\mu = \rho\frac{n\sqrt{\pi T}}{2}(0,1,\pm i,0\ldots)$. This is no further restriction since overall phases of α_n and $\tilde\alpha_n$ corresponds to shifts in $\sigma^\pm$ and the sign of α_n^2 corresponds to a choice of the X^2–direction. Then

$$X^\mu_{(\pm)} = \tfrac{\rho}{2}\Big(\tfrac{p^0\tau}{2\pi T},\ \mathrm{Re}(ie^{-in\sigma^+} + ie^{-in\sigma^-}),\ \mathrm{Re}(-e^{-in\sigma^+} \mp e^{-in\sigma^-}), 0\,, \ldots\Big) \tag{2.29}$$

$$= \tfrac{\rho}{2}\Big(\tfrac{p^0\tau}{2\pi T},\ \sin n\sigma^+ + \sin n\sigma^-,\ -\cos n\sigma^+ \mp \cos n\sigma^-, 0\,, \ldots\Big). \tag{2.30}$$

Using the formulas $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$, $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$, and $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$, we obtain

$$X^\mu_{(+)} = \rho\Big(\tfrac{p^0\tau}{4\pi T}, \sin\, n\tau\ \cos\, n\sigma, -\cos\, n\tau\ \cos\, n\sigma,\ 0, \ldots\Big), \tag{2.31}$$

$$X^\mu_{(-)} = \rho\Big(\tfrac{p^0\tau}{4\pi T},\ \sin\, n\tau\ \cos\, n\sigma,\ \sin\, n\tau\ \sin\, n\sigma,\ 0, \ldots\Big). \tag{2.32}$$

$X^\mu_{(+)}$ is a solution of total length $4n\rho$ and $2n\rho$ for closed and open strings, respectively. It corresponds to a rotating (multiply covered) rod of length 2ρ. According to (2.25) we have $p^0 = 2\rho n\,\pi T$, so that the ends indeed move with the speed of light (the tangential vector $\dot X^\mu_{(+)}$ is light-like at the boundary). The solution $X^\mu_{(-)}$ only exists for closed strings and corresponds to a periodically collapsing (multiply covered) circle of maximal radius ρ, i.e. maximal length $2\rho n\pi$. At the maximal radius there is no kinetic energy and we can check that $(mass)/(length) = T$ is the string tension. For open strings we always have kinetic energy and the factor of 2 in the string length matches the relative factor of 2 in the ratio $\sqrt{-p^2}/m$ for the two types of strings.

Evaluation of (2.17) shows that the angular momentum tensor decomposes into an orbit contribution $x_\mu P_\nu - x_\nu P_\mu$ and the left- and right-moving spin contributions Σ and $\tilde\Sigma$,

$$J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + \Sigma_{\mu\nu} + \tilde\Sigma_{\mu\nu}, \qquad \Sigma^{\mu\nu} = -i\sum_{n>0}\frac{1}{n}\left(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\right), \tag{2.33}$$

where $\tilde\Sigma$ should be omitted in case of open strings (in that case the σ integral only extends from 0 to π and the spin contribution has to be divided by 2). Inserting the above solutions we find

$$p^2 = (2\rho n\pi T)^2, \qquad \Sigma^{12} = \pm\tilde\Sigma^{12} = \rho^2 n\pi T. \tag{2.34}$$

The classical spin is given by $J = \Sigma^{12}$ and $J = \Sigma^{12} + \tilde\Sigma^{12}$ for open and closed strings, respectively. The length scale ρ drops out in the ratio J/m^2, whose maximal value is obtained for the lowest

[8] There are spurious solutions to (2.28) that are missed by the light cone gauge condition: Consider, for example, $p^\mu = 0$ and $\alpha^\mu_{\pm n} = \tilde\alpha^\mu_{\pm n} = (1,1,0,\ldots,0)$.

frequency $n=1$. This shows that the slope α' of the leading Regge trajectory is

$$\alpha'_{closed} = \frac{1}{4\pi T}, \qquad \alpha'_{open} = \frac{1}{2\pi T}, \qquad \frac{J}{m^2} \le \alpha'. \tag{2.35}$$

According to the literature it can be shown that all classical solutions obey this inequality. In the quantum theory it will be corrected by a constant shift α_0.

2.6 Poisson brackets and Virasoro algebra

As a first step towards quantization we now compute the Poisson brackets among the oscillators α and $\tilde\alpha$, which follow from the canonical brackets $\{\Pi^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{PB} = -\delta(\sigma-\sigma')\eta^{\mu\nu}$ with $\Pi^\mu = -T\dot X^\mu$, $\{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{PB} = 0$ and $\{\Pi^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\}_{PB} = 0$ by a Fourier analysis. $\delta(\sigma-\sigma')$ is understood to be 2π–periodic. Inserting (2.21) and (2.22) we find

$$\begin{aligned}\frac{\delta(\sigma-\sigma')\eta^{\mu\nu}}{T} &= \{\frac{p^\mu}{2\pi T} + \sum_{m\neq 0} \frac{\alpha^\mu_m e^{-im(\tau+\sigma)} + \tilde\alpha^\mu_m e^{-im(\tau-\sigma)}}{\sqrt{4\pi T}}, \\ &\qquad x^\nu + \frac{p^\nu \tau}{2\pi T} + \sum_{n\neq 0} \frac{i}{n} \frac{\alpha^\nu_n e^{-in(\tau+\sigma')} + \tilde\alpha^\nu_n e^{-in(\tau-\sigma')}}{\sqrt{4\pi T}}\}_{PB}.\end{aligned} \tag{2.36}$$

Since the variables p^μ, x^μ, α^μ_m and $\tilde\alpha^\mu_m$ parametrize the general solution to the equations of motions, general results of the canonical formalism tell us that we have to fulfill these relations at a fixed time, say $\tau = 0$. This fixes all brackets among the coefficients in the Fourier representation of $X^\mu(\tau,\sigma)$ and guarantees the canonical brackets for all times.

We first consider the closed string and pick out the brackets among the individual coefficients by evaluating the double integrals $\iint d\sigma\, d\sigma' e^{i(k\sigma + k'\sigma')}$. For $k = k' = 0$ we obtain

$$\{p^\mu, x^\nu\}_{PB} = \eta^{\mu\nu} \tag{2.37}$$

and $\{x^\mu, x^\nu\}_{PB} = \{p^\mu, p^\nu\}_{PB} = 0$. For $k = 0 \neq k'$ we obtain from the brackets $\{\dot X, \dot X\}_{PB}$ and $\{\dot X, X\}_{PB}$ at $\tau = 0$ that

$$\{p^\mu, \alpha^\nu_{k'} + \tilde\alpha^\nu_{-k'}\}_{PB} = 0 = \{p^\mu, \frac{1}{k'}\alpha^\nu_{k'} + \frac{1}{-k'}\tilde\alpha^\nu_{-k'}\}_{PB}, \tag{2.38}$$

hence $\{p^\mu, \alpha^\nu_n\}_{PB} = \{p^\mu, \tilde\alpha^\nu_n\}_{PB} = 0$. Similarly, for $k' = 0 \neq k$ the brackets $\{X, X\}_{PB}$ and $\{\dot X, X\}_{PB}$ imply that x^μ has vanishing brackets with all oscillators α and $\tilde\alpha$. Eventually, for k and k' non-zero we find that all brackets among α and $\tilde\alpha$ vanish and we conclude that

$$i\{\alpha^\mu_m, \alpha^\nu_n\}_{PB} = i\{\tilde\alpha^\mu_m, \tilde\alpha^\nu_n\}_{PB} = n\,\delta_{m+n}\,\eta^{\mu\nu}, \qquad i\{x^\mu, p^\nu\}_{PB} = -i\eta^{\mu\nu} \tag{2.39}$$

are the non-vanishing brackets, where δ_l is an abbreviation for $\delta_{l,0}$.

In case of open strings we must set $\tilde\alpha_n^\mu = \alpha_n^\mu$, which makes X^μ even and 2π periodic. It is convenient to integrate again over the interval $0 < \sigma, \sigma' < 2\pi$ in order that the exponentials provide a complete set of orthogonal vectors. But then we have to take into account a second contribution from the δ function, i.e. we must let $\delta(\sigma-\sigma') \to \delta(\sigma-\sigma')+\delta(\sigma+\sigma')$ in eq. (2.36). The double integral for the Fourier coefficients then gives $\frac{2\pi}{T}\eta^{\mu\nu}(\delta_{k-k'}+\delta_{k+k'})$ on the l.h.s. of that equation. We can now repeat the same calculation as above, with the only difference that the second δ function $\delta_{k+k'}$ now doubles the result for the bracket $\{x^\mu, p^\nu\}_{PB}$,

$$i\{x^\mu, p^\nu\}_{PB} = -2i\eta^{\mu\nu} \qquad \text{(open string).} \tag{2.40}$$

This can be understood easily because, with the ansatz (2.21), the total momentum P_μ is p_μ for closed strings and $p_\mu/2$ for open strings, so that $\{x_\mu, P_\nu\}_{PB} = -\eta_{\mu\nu}$ in both cases, as we should expect. $\delta_{k+k'}$ also allows for a non-vanishing bracket $\{\alpha_n, \tilde\alpha_{-n}\}_{PB}$, which is necessary because of the identification of $\tilde\alpha$ and α. Otherwise the Poisson brackets are the same for open and closed strings.

Recall that the Virasoro generators $L_n := T\int_0^{2\pi} d\sigma^+ T_{++}e^{in\sigma^+}$ and $\tilde L_n := T\int_0^{2\pi} d\sigma^- T_{--}e^{in\sigma^-}$, which are the Fourier modes of the energy momentum tensor, are given by

$$L_n = -\frac{1}{2}\sum_{m=-\infty}^{\infty} \eta_{\mu\nu}\alpha_m^\mu \alpha_{n-m}^\nu, \qquad \tilde L_n = -\frac{1}{2}\sum_{m=-\infty}^{\infty} \eta_{\mu\nu}\tilde\alpha_m^\mu \tilde\alpha_{n-m}^\nu. \tag{2.41}$$

They satisfy an infinite-dimensional Lie algebra, the **Virasoro algebra**

$$i\{L_m, L_n\}_{PB} = (m-n)L_{m+n}, \qquad i\{\tilde L_m, \tilde L_n\}_{PB} = (m-n)\tilde L_{m+n}, \tag{2.42}$$

as is easily verified using that $i\{L_n, \alpha_l^\rho\}_{PB} = -l\alpha_{n+l}^\rho$.

A more direct way to understand this algebra is to observe that any *conformal* Killing vector field ξ defines a conserved current $J_\xi^m = \sqrt{-g}\,\xi^l T_l{}^m$ if the energy–momentum tensor is traceless and conserved (which can be shown to be consequences of Weyl invariance and the equations of motion of the matter fields, respectively). For left-moving conformal reparametrizations with $\xi = \xi^+(\sigma^+)\partial_+$ the corresponding conserved quantity is $L_\xi := \int d\sigma\, \xi^+ T_+{}^0 = \int d\sigma^+ \xi^+ (T_+{}^+ + T_+{}^-)/2 = \int d\sigma^+ \xi^+ T_{++}$ (recall that $\eta^{+-} = 2$; $d\sigma$ can be replaced by $d\sigma^+$ because the integrand only depends on $\sigma^+ = \sigma + \tau$). If we choose the basis $\xi_n = e^{in\sigma^+}\partial_+$ for periodic infinitesimal reparametrizations of σ^+, we find the Lie brackets $[\xi_m, \xi_n] = [e^{im\sigma^+}\partial_+, e^{in\sigma^+}\partial_+] = i(n-m)e^{i(m+n)\sigma^+}\partial_+$. Since L_ξ generates the Lie derivative $\{L_\xi, X\}_{PB} = -\xi^m\partial_m X$, and because of the Jacobi identity, the Poisson algebra of the charges $L_n = L_{\xi_n}$ must have the same structure constants. Note that the Virasoro constraints L_n and $\tilde L_n$ are conserved quantities, i.e. it is sufficient to impose $T_{mn} = 0$ at some initial time. Obviously, the conformal algebra in two dimensions is the direct product of two identical, infinite dimensional Lie algebras.

Chapter 3

Quantization of bosonic strings

Before actually performing the quantization in section 2.1 let us discuss some general aspects of canonical quantization of gauge invariant systems. It is clear by now that the 'matter fields' X^μ are the dynamical fields of the bosonic string, and that, at least locally, the metric only consists of gauge degrees of freedom. In the Hamiltonian formalism this is indicated by the fact that the conjugate momenta $(\pi_g)^{mn} = \partial L/\partial \dot g_{mn}$ vanish identically. Therefore we cannot naively impose the Poisson brackets, at least not as a 'strong' identity: In the process of quantization it is certainly not consistent to impose a commutator $[g_{mn}, (\pi_g)^{kl}] = i(\delta^k_m\delta^l_n + \delta^k_n\delta^l_m)/2$ if $(\pi_g)^{kl} \equiv 0$.

Dirac and Bergmann [DI64] developed a method for obtaining a Hamiltonian description if the Legendre transformation is singular (as it happens in our case): The defining equations for the momenta cannot be solved for the time derivatives of the coordinates *iff* there are relations among the coordinates and momenta, the *primary constraints* $\Phi_{\hat I}(p, q) = 0$. If this happens, then the Hamiltonian is only defined up to terms proportional to the constraints. These must be fulfilled at all times, so we must have $\{\Phi_{\hat I}, H\}_{PB} = 0$ (here we ignore the fact that the coordinates and momenta are constrained and compute the naive PB). If there is no choice for the Hamiltonian that makes the l.h.s. of this equation proportional to the $\Phi_{\hat I}$, then we get additional constraints, which we call *secondary*. (In our case the secondary constraint $\{(\pi_g)^{kl}, H\}_{PB} = 0$ is equivalent to the vanishing of the energy momentum tensor.)

After all constraints $\Phi_I = 0$, primary and secondary, are known, we have to calculate their Poisson algebra: If the antisymmetric matrix $c_{IJ} = \{\Phi_I, \Phi_J\}_{PB}$ vanishes on the constraint surface $\Phi_I = 0$, i.e. if it is a linear combination of constraints $\{\Phi_I, \Phi_J\}_{PB} = f_{IJ}{}^K\Phi_K$, then the Φ_I are called *first class*. This type of constraints indicates gauge symmetries (the Virasoro algebra is an example). Indeed, in case of gauge symmetries the equations of motion cannot be of the form $\dot f = \{f, H\}_{PB}$ with a *unique* Hamiltonian, because the time evolution is not fixed by the Euler–Lagrange equations of motion.[1] For quantization it is necessary to get rid of the

[1] The full gauge freedom is recovered in the form of an arbitrary linear combination of the constraints $\sum \lambda^I \Phi_I$

first class constraints by imposing additional gauge fixing constraints (this leads to the 'reduced phase space'). One problem of the Dirac procedure is that the set of available gauge fixings is too restricted. In the case of QED, for example, we cannot impose a Lorentz covariant gauge, simply because the time derivative of A_0 is not available. This shortcoming has been cured by the BFV formalism [fr75, HE92]: The phase space is extended with dynamical Lagrange multipliers and ghosts, making available, among other benefits, covariant gauge fixings. We will, however, arrive at the same result with the following short cut: We fix the gauge before we perform the Legendre transformation, and thus never get first class constraints. In order to compensate the resulting propagation of unphysical degrees of freedom we introduce ghosts, and the cancellation of all unphysical contributions to physical quantities is controlled by BRST invariance (see below).

If $c_{IJ} = \{\Phi_I, \Phi_J\}_{PB}$ has maximal rank on the constraint surface $\Phi_I = 0$ then the Φ_I are called *second class*. This type of constraints are caused by having too many degrees of freedom: Usually the Lagrangian is quadratic in the time derivatives so that the equations of motion are second order, i.e. two functions have to be specified on a Cauchy surface (the fields and their time derivatives). The Hamiltonian equations of motion, on the other hand, are first order, but now we have twice as many 'off shell' degrees of freedom (for each coordinate we introduce a momentum). This counting is spoiled if the Lagrangian is only linear in the time derivatives. Accordingly, $p_i = \partial L/\partial \dot{q}^i$ is a function of the coordinates only, and, instead of defining $\dot{q}^i$ in terms of phase space variables, this equation is a second class constraint (as c_{IJ} is antisymmetric and invertible such constraints must occur in pairs).

So what happens is that we introduce too many phase space variables and that the redundant momenta can be eliminated by the constraint equations. Accordingly, the PBs have to be replaced by the Dirac brackets, which only take into account the true degrees of freedom: $\{f,g\}_{DB} = \{f,g\}_{PB} - \{f,\Phi_I\}_{PB}\, c^{IJ}\, \{\Phi_J, g\}_{PB}$ with $c_{IJ} = \{\Phi_I, \Phi_J\}_{PB}$ and $c_{IJ}c^{JK} = \delta_I^K$, so that $\{f, \Phi_I\}_{DB} = 0$ for all functions f on phase space and for all constraints Φ_I. But we will again use a short cut. Consider the inverse Legendre transformation and the resulting variational equations,

$$L = \dot{q}^i p_i - H(p,q), \qquad \frac{\delta L}{\delta q^i} = -\dot{p}_i - \frac{\partial H}{\partial q^i} = 0 \qquad \frac{\delta L}{\delta p_i} = (-)^i \dot{q}^i - \frac{\partial H}{\partial p_i} = 0. \tag{3.1}$$

If we do not eliminate the momenta from their variational equation we always find the situation with second class constraints (this is sometimes called *first order formalism*). For a Lagrangian of this form we can therefore directly read off the conjugate pairs of phase space variables and introduce the corresponding brackets, instead of introducing momenta for q^i and for p_i and eliminating them in a second step with the Dirac procedure.

that can be added to the Hamiltonian. This is know as Dirac's conjecture and has been proven under certain regularity assumptions. The coefficients λ^I are called Lagrange multiplier (fields).

The final ingredient that we need for quantization is the BRST formalism [be76]. The introduction of 'ghost fields' that have the same quantum numbers – including spin – as the gauge degrees of freedom, but the opposite statistics, was suggested already in 1963 by Feynman in the context of quantum gravity [fe63]. This was motivated by the observation that the gauge degrees of freedom propagate after gauge fixing and that their contribution to loop diagrams is not transversal, like in QED, and therefore spoils unitarity and gauge independence. In 1967 Faddeev and Popov used ghosts to bring the field dependent functional determinant that arises from gauge fixing in non-abelian gauge theories into the exponent of the path integral [IT80].

Later it was observed by Becchi, Rouet and Stora [be76] that the resulting action has a global fermionic nilpotent symmetry $s^2 = 0$ with $s\phi^i = c^I\delta_I\phi^i$, i.e. for a δ_I-invariant action the BRST transformation of a matter field ϕ^i is equal to its gauge transformation with the gauge parameters replaced by ghost fields. In order to have a well-defined ghost number that is consistent with the dynamics of the ghosts, we also need to introduce anti-ghost fields $\bar{c}^I$ with ghost number -1, whose BRST transform $b^I := s\bar{c}^I$ is usually called *lagrange multiplier field.* For a general gauge theory with an irreducible closed gauge algebra $[\delta_I, \delta_J] = \mathcal{F}_{IJ}{}^K\delta_K$ it can be shown that $s^2\phi^i = 0$ implies that the BRST transformation of the ghosts is

$$sc^K = \tfrac{(-)^I}{2}\, c^I c^J \mathcal{F}_{JI}{}^K. \tag{3.2}$$

Nilpotency of s, i.e. the equation $s^2c^I = 0$, is then equivalent to the Jacobi identity

$$\sum_{IJK}(-)^{IK}\left(\delta_I\mathcal{F}_{JK}{}^L + \mathcal{F}_{IJ}{}^M\mathcal{F}_{MK}{}^L\right) = 0, \tag{3.3}$$

which follows from $\sum_{IJK}(-)^{IK}[[\delta_I,\delta_J],\delta_K] = 0$. (We have an *open* gauge algebra if the graded commutator $[\delta_I,\delta_J]$ is proportional to δ_K only on shell, i.e. up to the equations of motion. In that case on needs the BV antibracket formalism [ba81]. Irreducibility of the gauge algebra means that the gauge transformations δ_I are linearly independent.) The BRST algebra thus encodes the structure of the symmetry algebra in a very efficient way.

The role of the BRST symmetry in canonical quantization was eventually clarified by Kugo and Ojima [ku79]: Initially it was assumed that $\bar{c}$ is the complex conjugate of c, but then gauge fixing does not give a real Hamiltonian and thus formally spoils unitarity. Rather, c and $i\bar{c}$ are independent real fields.[2] In the quantum theory the conserved charge Q_{BRST} that corresponds to the BRST symmetry commutes with the Hamiltonian (up to anomalies), so it can be used to define a 'physical' subspace of the Fock space with the physical states defined by the condition $Q|phys\rangle = 0$, which is consistent with time evolution.

If there is no anomaly in the commutation relation $\{Q,Q\} = 2Q^2 = 0$ then all states fall into doublet and singlet representations $(|\psi\rangle, |Q\psi\rangle)$ and $|\psi_{phys}\rangle$ of Q. For the doublets the dual

[2] With this assignment gauge transforms $s\phi^i$ of real fields, lagrange multiplier fields $s\bar{c}$, and gauge fixing terms $s\psi$ with imaginary anticommuting ψ are real. Note that $(XY)^* = (-)^{XY}X^*Y^*$ and $\mathcal{O}^*\phi = (-)^{\mathcal{O}\phi}(\mathcal{O}\phi^*)^*$.

states must also form a doublet since the BRST charge Q, which generates a real symmetry transformation, should be hermitian. Furthermore, BRST-trivial states $Q|\psi\rangle$ have vanishing scalar product with all physical states, which therefore correspond to cohomology classes of Q-invariant states modulo Q-exact states. This is the 'quartet mechanism' by which doublet states cannot contribute to negative norm states in the physical Hilbert space. What remains to be checked for a given theory is that the 'physical Hilbert space' that we end up in this way does not contain any negative norm states.

Expectation values of physical operators, i.e. observables, should not depend on the representative we choose for a physical state. This is guaranteed if $\mathcal{O}$ (anti)commutes with Q, i.e. $[Q, \mathcal{O}_{phys}] = 0$. In turn, physical expectation values of Q-exact operators $\mathcal{O} = [Q, \mathcal{O}']$ vanish, so that physical obervables also correspond to cohomology classes. In particular, the sum of gauge dependent and ghost dependent terms of an s-invariant classical action with vanishing ghost number can be shown to be s-exact: $\mathcal{L}(\phi, c, \bar{c}, b) = \mathcal{L}_{inv.}(\phi) + s\Psi(\phi, c, \bar{c}, b)$ (ψ is often called *gauge fermion*). This suggests that physical quantities should be independent of the choice of the gauge fixing term $s\psi$, which is known as the Fradkin–Vilkovisky theorem [HE92].

3.1 BRST quantization

Now we are ready to apply the above machinery to the case of the bosonic string. The Polyakov action is invariant under the nilpotent transformation

$$sX^\mu = c^l\partial_l X^\mu, \qquad sg_{mn} = D_m c_n + D_n c_m - 2g_{mn}\lambda, \qquad sc^m = c^l\partial_l c^m, \qquad s\lambda = c^l\partial_l\lambda, \tag{3.4}$$

where c^m are the diffeomorphism ghosts and λ is the Weyl ghost. We want to fix the metric to a background value $\hat{g}_{mn}$, which we initially keep arbitrary. We will see that the equations of motion for the antighost field b_{mn} imply that b_{++} is a function of σ^+, so that this field naturally has lower indices. It is thus convenient to fix the inverse metric $g^{mn}(\sigma) = \hat{g}^{mn}(\sigma)$ and we add the gauge fixing and ghost term $\int d^2\sigma\mathcal{L}^{(c)}$ with

$$\tfrac{2}{T}\mathcal{L}^{(c)} = s(\sqrt{-g}\, b_{mn}(g^{mn} - \hat{g}^{mn})) = \tilde{L}_{mn}(g^{mn} - \hat{g}^{mn}) + 2\sqrt{-g}\, b_{mn}(g^{ml}D_l c^n - g^{mn}\lambda), \tag{3.5}$$

and $\tilde{L}_{mn} = \sqrt{-g}\, L_{mn} = s(\sqrt{-g}\, b_{mn})$ to the Polyakov action. Note that the quantum numbers of the anti-ghost field come from the gauge fixing term, whereas those of the ghosts are inherited from the gauge transformation (only the numbers of degrees of freedom must coincide for ghosts and anti-ghosts, but quantum numbers like the spin can be different). The factor $\sqrt{-g}$ is inserted to make b_{mn} a symmetric tensor rather than a tensor density.

Variation with respect to g^{mn}, L_{mn}, λ and b_{mn} implies the equations of motion

$$L_{mn} + T^{(X)}_{mn} + T^{(c)}_{mn} = 0, \qquad g^{mn} = \hat{g}^{mn}, \qquad b_{mn}g^{mn} = 0, \qquad 2\lambda = D_n c^n, \tag{3.6}$$

which are algebraic for the fields L_{mn}, g^{mn}, λ and for the trace of the anti-ghost.

$$T^{(c)}_{mn} = (b_{mj}D_nc^j + b_{nj}D_mc^j + D_j(b_{mn}c^j) - g_{mn}g^{ij}b_{jk}D_ic^k) + (g_{mn}g^{ij}b_{ij} - 2b_{mn})\lambda \tag{3.7}$$
$$= b_{mj}D_nc^j + b_{nj}D_mc^j + D_jb_{mn}c^j - g_{mn}g^{ij}b_{jk}D_ic^k \tag{3.8}$$

is the ghost contribution to the energy–momentum tensor[3] and $T^{(X)}_{mn}$ is the 'matter' contribution coming from the Polyakov action. $g^{mn}b_{mn} = 0$ implies that $T^{(c)}_{mn}$ is traceless. Furthermore, the total energy–momentum $T_{mn} = T^{(X)}_{mn} + T^{(c)}_{mn}$, corresponding to the action

$$\mathcal{L} = \mathcal{L}_P + T\sqrt{-g}\,b_{mn}(g^{ml}D_lc^n - \tfrac{1}{2}g^{mn}D_lc^l), \tag{3.9}$$

is proportional to the BRST variation of the traceless anti-ghost b_{mn}. Note that we can eliminate a set of fields whose *own* equations of motion are algebraic by inserting their values back into the action.

In light-cone coordinates we find $T^{(c)}_{+-} = 0$ and $T^{(c)}_{++} = 2b_{++}D_+c^+ + D_+b_{++}c^+$, where we used the equation of motion $\delta S/\delta c^n = g^{ml}D_lb_{mn} = 0$, implying $D_-b_{++}c^- = 0$. The Christoffel symbols drop out of this expression so that

$$T^{(c)}_{++} = 2b_{++}\partial_+c^+ + \partial_+b_{++}c^+, \qquad T^{(c)}_{--} = 2b_{--}\partial_-c^- + \partial_-b_{--}c^-, \tag{3.10}$$

Since $\hat\Gamma_{++}{}^+$ and $\hat\Gamma_{--}{}^-$ are the only non-vanishing components of the Christoffel symbol and $\sqrt{-g}\,g^{+-} = 1$, the complete Lagrangian in light-cone coordinates is

$$\mathcal{L} = T\partial_+X^\mu\partial_-X^\nu G_{\mu\nu} + T(b_{++}\partial_-c^+ + b_{--}\partial_+c^-) \tag{3.11}$$

and the equations of motion imply that b_{++} and c^+ only depend on σ^+.

From (3.9) it follows that the imaginary field $-Tb_{++}$ is the conjugate momentum to c^+,

$$i\{b_{++}(\tau+\sigma), c^+(\tau+\sigma')\}_{PB} = \tfrac{i}{T}\delta(\sigma-\sigma') = i\{b_{--}(\tau-\sigma), c^-(\tau-\sigma')\}_{PB}. \tag{3.12}$$

For the Fourier modes $b_n = (b_{-n})^\dagger = -iT\int d\sigma\, b_{++}e^{in\sigma^+}$ and $c_n = (c_{-n})^\dagger = \frac{1}{2\pi}\int d\sigma\, c^+e^{in\sigma^+}$, and their right-moving relatives with

$$b_{--} = \frac{i}{2\pi T}\sum_{n=-\infty}^{\infty}\tilde b_n e^{-in\sigma^-}, \qquad c^- = \sum_{n=-\infty}^{\infty}\tilde c_n e^{-in\sigma^-}, \tag{3.13}$$

this implies the Poisson brackets

$$i\{b_n, c_m\}_{PB} = \delta_{m+n} = i\{\tilde b_n, \tilde c_m\}_{PB} \tag{3.14}$$

[3] The most tedious part of the computation of $T^{(c)}_{mn} = \frac{1}{T}\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}^{(c)}}{\delta g^{mn}}$ is the variation of the Christoffel symbol contained in the covariant derivative: $\delta(D_lc^n) = \delta\hat\Gamma_{lm}{}^nc^m = \frac{1}{2}c^m\delta(g^{nk}(\partial_lg_{mk} + \partial_mg_{lk} - \partial_kg_{ml}))$. Since both sides of this equation are covariant, all terms linear in Christoffel symbols or partial derivatives of the metric must cancel and we immediately obtain $\delta(D_lc^n) = \frac{1}{2}g^{nk}c^m(D_l\delta g_{mk} + D_m\delta g_{lk} - D_k\delta g_{ml})$. Covariant partial integration of the variation of the action then leads to (3.7).

which have to be replaced by anticommutators for the quantized oscillator modes.

In light-cone coordinates the BRST current $J_S^m = -sX^\mu \frac{\partial\mathcal{L}}{\partial\partial_m X^\mu} - sc^l \frac{\partial\mathcal{L}}{\partial D_m c^l} + c^m \mathcal{L}_P$ reads

$$\begin{aligned} J_S^- &= -2Tc^l\partial_l X^\mu\ \partial_+ X^\nu G_{\mu\nu} - 2Tc^n\partial_n c^+\ b_{++} + 2Tc^-\ \partial_+ X^\mu \partial_- X^\nu G_{\mu\nu} && (3.15)\\ &= -2T(c^+\partial_+ X^\mu \partial_+ X^\nu G_{\mu\nu} + b_{++}c^+\partial_+ c^+). && (3.16)\end{aligned}$$

This suggests to define left- and right-moving BRST charges $Q_\pm$ with $Q = -\int d\sigma\, J_S^0 = -\frac{1}{2}(\int d\sigma^+ J_S^- + \int d\sigma^- J_S^+) = Q_+ + Q_-$,

$$Q_+ = T\int d\sigma^+\ c^+(\partial_+ X^\mu \partial_+ X^\nu G_{\mu\nu} + b_{++}\partial_+ c^+) = T\int d\sigma^+\ c^+(T_{++}^{(X)} + \frac{1}{2}T_{++}^{(c)}) \qquad (3.17)$$

and its right-moving partner Q_-; note that $(c^+)^2 = 0$ and $c^+D_+c^+ = c^+\partial_+ c^+$.

For a flat target space we can insert the solutions to the equations of motion. For the Fourier modes L_n of $T_{++} = \frac{1}{2\pi T}\sum L_n e^{-in\sigma^+}$ we obtain

$$L_n^{(X)} = -\tfrac{1}{2}\sum_{m=-\infty}^{\infty} :\alpha_{n-m}\cdot\alpha_m:, \qquad L_n^{(c)} = \sum_{m=-\infty}^{\infty}(n+m):b_{n-m}c_m: \qquad (3.18)$$

and for the BRST charge

$$\begin{aligned} Q_+ &= \sum_{n=-\infty}^{\infty} :(L_n^{(X)} + \frac{1}{2}L_n^{(c)})c_{-n}: - ac_0 && (3.19)\\ &= \sum_{n=-\infty}^{\infty} L_n^{(X)}\ c_{-n} - \frac{1}{2}\sum_{n,m=-\infty}^{\infty}(m-n):c_{-m}c_{-n}b_{m+n}: - ac_0, && (3.20)\end{aligned}$$

where we introduced normal ordering symbol :: that puts creation operators (negative index) to the left and a coefficient a parametrizing the ordering ambiguity in Q_+. We find

$$[L_n, b_l] = (n-l)b_{n+l}, \qquad [L_n, c_l] = -(2n+l)c_{n+l} \qquad (3.21)$$

with $\{Q_+, b_n\} = L_n := L_n^{(X)} + L_n^{(c)} - a\delta_n$, and $\{Q_+, c_l\} = \sum_n\ (n+l/2)c_{l+n}c_{-n}$.

Next we turn to the construction of a Fock space representation of our operator algebra. Recall the commutation relations

$$[\alpha_m^\mu, \alpha_n^\nu] = -m\delta_{m+n}\eta^{\mu\nu}, \qquad [x^\nu, P^\mu] = -i\eta^{\mu\nu}, \qquad \{b_m, c_n\} = \delta_{m+n}. \qquad (3.22)$$

with $p^\mu = \sqrt{4\pi T}\alpha_0^\mu$. We define a vacuum state that is annihilated by all oscillators with positive mode number, i.e. $\alpha_n^\mu|0\rangle = b_n^\mu|0\rangle = c_n^\mu|0\rangle = 0$ for $n > 0$. The difficult part is the treatment of the zero modes. All states can be constructed as sums of tensor products of a coordinate factor and a ghost factor. For the bosonic part we can, for example, diagonalize the momentum and define eigenstates $P_\mu|k\rangle = k_\mu|k\rangle$, so that $|k\rangle =\ :e^{ikX}:\ |0\rangle$. In the ghost sector the zero mode algebra is $b_0^2 = c_0^2 = 0$ and $\{b_0, c_0\} = 1$. We cannot diagonalize a nilpotent operator, so we need to introduce a 2-dimensional representation space with

$$\begin{array}{lll} b_0|\uparrow\rangle = |\downarrow\rangle & c_0|\uparrow\rangle = 0 & \langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 1 \\ b_0|\downarrow\rangle = 0 & c_0|\downarrow\rangle = |\uparrow\rangle & \langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 0 \end{array} \tag{3.23}$$

and $\langle\downarrow|c_0 = \langle\uparrow|$, $\langle\uparrow|b_0 = \langle\downarrow|$, $\langle\uparrow|b_0|\uparrow\rangle = 1 = \langle\downarrow|c_0|\downarrow\rangle$.

3.2 Conformal anomaly and critical dimension

For the consistency of the BRST quantization program we have to check that $Q^2 = 0$. This will fix the constant a in eq. (3.20) and also gives us the critical dimension. First we observe that $Q_+^2 = 0$ implies that the Virasoro algebra has no anomalous contribution (the anticommutator of Q_+ and Q_- vanishes trivially, so we only need to consider left movers). Indeed, $0 = [\{Q_+, Q_+\}, b_n] = [Q_+, \{Q_+, b_n\}] - [\{Q_+, b_n\}, Q_+] = 2[Q_+, L_n]$, hence

$$[L_m, L_n] = [L_m, \{Q_+, b_n\}] = \{Q_+, [L_m, b_n]\} = (m-n)\{Q_+, b_{m+n}\} = (m-n)L_{m+n}. \tag{3.24}$$

The converse is also true since one can show that

$$Q_+^2 = \frac{1}{2}\sum c_{-m}c_{-n}([L_m, L_n] - (m-n)L_{m+n}). \tag{3.25}$$

This calculation, however, is very tedious, so we postpone it till we have more efficient tools for computing commutators of normal ordered expressions when we come to operator products and contour integrals in the complex plane.

In any case, absence of anomalies in the Virasoro algebra is necessary for $Q^2 = 0$. Since L_0 is the only mode for which there is an ordering ambiguity it is easy to see that

$$[L_m, L_n] = (m-n)L_{m+n} + A_m\delta_{m+n}. \tag{3.26}$$

Obviously, $A_{-m} = -A_m$ and $A_0 = 0$. From the Jacobi identity $\sum_{lmn}[L_l, [L_m, L_n]] = 0$ if follows for $l + m + n = 0$ that

$$(m-n)A_l + (n-l)A_m + (l-m)A_n = 0. \tag{3.27}$$

For $l = 1$ we get $(n-1)A_{n+1} = (n+2)A_n - (2n+1)A_1$ which determines all A_m in terms of A_1 and A_2. Since $A_m = m$ and $A_m = m^3$ solve this equation, we find

$$A_m = \tfrac{A_2-2A_1}{6}\, m^3 - \tfrac{A_2-8A_1}{6}\, m. \tag{3.28}$$

The final step in the calculation of the anomaly is to fix the two remaining constants by evaluating expectation values $A_m = \langle\uparrow|\, L_mL_{-m} - 2mL_0\, |\downarrow\rangle$ of (3.26) where $m > 0$.

$$A_1 \;=\; \langle\uparrow|\, (\alpha_0\cdot\alpha_1)(\alpha_0\cdot\alpha_{-1}) - 2(-\tfrac{\alpha_0^2}{2} - a) + (b_1c_0 + 2b_0c_1)(-b_{-1}c_0 - 2b_0c_{-1})\, |\downarrow\rangle$$

$$= \langle \uparrow | \, 2a - (2b_0c_1)(b_{-1}c_0) \, | \downarrow \rangle = 2a - 2, \tag{3.29}$$

$$A_2 = \langle \uparrow | \, (\tfrac{\alpha_1^2}{2} + \alpha_0 \cdot \alpha_2)(\tfrac{\alpha_{-1}^2}{2} + \alpha_0 \cdot \alpha_{-2}) + 4(\tfrac{\alpha_0^2}{2} + a) \, | \downarrow \rangle$$

$$- \langle \uparrow | \, (2b_2c_0 + 3b_1c_1 + 4b_0c_2)(2b_{-2}c_0 + 3b_{-1}c_{-1} + 4b_0c_{-2}) \, | \downarrow \rangle$$

$$= \langle \uparrow | \, \tfrac{1}{4}\alpha_1^2\alpha_{-1}^2 + 4a - (3 \cdot 3 + 4 \cdot 2) \, | \downarrow \rangle = D/2 + 4a - 17. \tag{3.30}$$

Putting the pieces together we find

$$A_m = \frac{m^3}{12}(D - 26) - \frac{m}{12}(D - 2 - 24a), \tag{3.31}$$

so that $Q^2 = 0$ implies $a = 1$ and $D = 26$. Note that the term linear in m depends on a. This can be used to eliminate A_1 even if $Q^2 \neq 0$, i.e. to bring the anomaly into the form

$$A_m = \frac{m^3 - m}{12} c, \tag{3.32}$$

where c is called central charge (in our case $a = 1$ is the appropriate value for any D). Then the $SL(2)$ subalgebra of the Virasoro algebra that is generated by L_0 and $L_{\pm 1}$ is free of anomalies[4] (the non-vanishing commutators are $[L_{\pm 1}, L_0] = \pm L_{\pm 1}$ and $[L_1, L_{-1}] = 2L_0$).

In addition to the BRST charge there is another (classically) conserved quantity: We assign ghost number ± 1 to ghosts c^m, λ and antighosts b_{mn}, respectively, and observe that our BRST-invariant classical action has ghost number 0. It is thus invariant under the infinitesimal transformation $\delta c^n = c^n$, $\delta \lambda = \lambda$ and $\delta b_{mn} = -b_{mn}$. This leads to the conserved Noether current $J_{gh}^m = -\delta c^n \frac{\partial \mathcal{L}}{\partial \partial_m c^n} = T\sqrt{-g}\, g^{ml} b_{ln} c^n$, which again simplifies nicely in the light cone gauge: $J_{gh}^{+} = 2Tb_{--}c^{-}$. There is thus a left-moving and a right-moving contribution to the ghost number, $\mathcal{N} = -i \int d\sigma J_{gh}^0 = -\frac{i}{2} \int d\sigma (J_{gh}^{+} + J_{gh}^{-}) = \mathcal{N}_+ + \mathcal{N}_-$,

$$\mathcal{N}_+ = \int \tfrac{d\sigma^+}{2i} J_{gh}^{-} = \sum_{n=-\infty}^{\infty} : c_n b_{-n} : \; + \text{const.} = \tfrac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{n>0} (c_{-n} b_n - b_{-n} c_n) + \tfrac{3}{2}, \tag{3.33}$$

where we include a factor of i in the definition of the charge to make the eigenvalues real. The reason for our asymmetric choice of the constant coming from the operator ordering ambiguity will become clear below. It leads to $\mathcal{N}_+ | \uparrow \rangle = 2 | \uparrow \rangle$ and $\mathcal{N}_+ | \downarrow \rangle = | \downarrow \rangle$. We will see later that J_{gh} is not conserved in the quantum theory. An anomaly of a global symmetry, however, does not spoil the consistency of a theory; the anomalous violation of ghost number conservation is, in fact, related to the topology of the world sheet and will play an important role in interactions.

[4] This is a special case of the following result: An anomalous term like the one in eq. (3.26) is called a central extension if it is consistent with the Jacobi identity. It is easy to show that semi-simple (finite-dimensional) Lie algebras only admit trivial central extensions, i.e. the 'central' terms in the algebra can be eliminated by adding constants to the generators.

3.3 Physical states

The physical subspace of our Fock space is defined by the cohomology of Q. We first consider states of the form $P(\alpha)|k\rangle\otimes|\uparrow\rangle$ or $P(\alpha)|k\rangle\otimes|\downarrow\rangle$, where $P(\alpha)$ is a polynomial in the physical creation operators α^μ_{-m}. For such states

$$Q_+\left(P(\alpha)|k\rangle\otimes|\downarrow\rangle\right) = \sum_{n\geq 0}\left(L_n^{(X)}-\delta_{n,0}\right)P(\alpha)|k\rangle\otimes c_{-n}|\downarrow\rangle, \tag{3.34}$$

$$Q_+\left(P(\alpha)|k\rangle\otimes|\uparrow\rangle\right) = \sum_{n>0} L_n^{(X)}P(\alpha)|k\rangle\otimes c_{-n}|\uparrow\rangle. \tag{3.35}$$

This looks similar to Gupta–Bleuler in QED, where the annihilation part of the gauge condition is imposed as a constraint on physical states. In the present context we need to make sure that physical expectation values of T_{++} vanish. Since the states built on $|\downarrow\rangle$ are dual to the states built on $|\uparrow\rangle$ the above formulas imply that all expectation values of L_n between physical states that do not contain ghost or antighost creation operators vanish (it is, of course, true in general, since $L_n=\{Q,b_n\}$). Hence our formalism reduces to the 'old covariant approach' [GR87] in the ghost-free sector.

Since the mass shell operator $L_0=\{Q,b_0\}$, the momentum operator P^μ and Q all commute with one another, we can compute the cohomology for Q for fixed eigenspaces with eigenvalues. If $(L_0-\lambda)|\Phi\rangle=0$ with $\lambda\neq 0$ for some Q-invariant state $|\Phi\rangle$ then $|\Phi\rangle=Q(\frac{1}{\lambda}b_0|\Phi\rangle)$ is Q-exact, so that non-trivial physical states must be on-shell states. In this way we recover the mass shell condition $L_0=0$ also for the states of the form (3.35) that are built on $|\uparrow\rangle$. Moreover, it can be shown that representatives of all physical states can be chosen to be of the form (3.34) or of the form (3.35) [th89]; there is a one-to-one correspondence of these states, which can be obtained from one another by application of b_0 or c_0. So we have a two-fold degeneracy, which follows from the existence of ghost zero modes and from the 'quartet mechanism' [ku79], i.e. the fact that dual states of BRST singlets and doublets form singlets and doublets, respectively.

This can be used to give a simple proof of the fact that the ghosts drop out of the cohomology, except for their zero modes: Since $\{Q,b_n^\dagger\}=L_n^\dagger$, standard arguments of homological algebra show that all $b_n^\dagger$ with $n>0$ drop out of the cohomology if the $L_n^\dagger$ can be used as part of a basis for the algebra of creation operators, which can be shown to be true if $P^\mu\neq 0$. For $P^\mu=0$ there are only $2D+4$ on-shell states, which are the singlets $b_{-1}|\downarrow\rangle$, $\alpha^\mu_{-1}|\downarrow\rangle$ and their dual states $c_{-1}|\uparrow\rangle$, $\alpha^\mu_{-1}|\uparrow\rangle$, and the self-dual doublet $b_{-1}|\uparrow\rangle$ and $Q(b_{-1}|\uparrow\rangle)=-2c_{-1}|\downarrow\rangle$. The '$SL(2,\mathbb{C})$ invariant vacuum' $|0\rangle=b_{-1}|\downarrow\rangle$ and its dual are the only Lorentz-invariant physical states. It will play an important role in conformal field theory.

Since the physical states that are built on $|\downarrow\rangle$ are automatically on-shell, whereas those on $|\uparrow\rangle$ are off-shell null states or on-shell limits of null states, the states $P(\alpha)|\downarrow\rangle$ seem to be somewhat preferable. So we may choose the 'Siegel gauge' $b_0|\Phi\rangle = 0$ in addition to the physical state condition $Q|\Phi\rangle = 0$, a relaxed form of which plays an important role in closed string field theory [zw93, be94, ne89, di$_{\rm b}$91].

It has been shown a long time ago that in $D = 26$ dimensions all physical states can be generated from tachyonic vacua $|k\rangle$ with $\frac{1}{4\pi T}k^2 = -2$ by repeated application of the so-called DDF creation operators $(A^i_m)^\dagger = A^i_{-m}$ with $m > 0$ [de72, GR87], which are zero modes of 'transversal vertex operators for massless states' and satisfy

$$[A^i_m, A^j_n] = m\delta_{ij}\delta_{m+n}, \qquad A^i_m = \int \frac{d\tau}{2\pi}\varepsilon^i_\mu \dot{X}^\mu e^{imqX}, \tag{3.36}$$

where $q^2 = 0$, $qk = 1$, $\varepsilon^i k = \varepsilon^i q = 0$ and $\varepsilon^i\varepsilon^j = -\delta^{ij}$. This implies that all physical states have positive norm.

We now consider the tachyon and the massless states in more detail. Since $[L^{(X)}_m, L^{(X)}_1] = (m-1)L^{(X)}_{m+1}$ for $m > 0$ it is sufficient to impose

$$L^{(X)}_0 = -(\frac{1}{2}\alpha_0^2 + \alpha_{-1}\cdot\alpha_1 + \alpha_{-2}\cdot\alpha_2 + \ldots) = 1, \tag{3.37}$$

$$L^{(X)}_1 = -(\alpha_0\cdot\alpha_1 + \alpha_{-1}\cdot\alpha_2 + \alpha_{-2}\cdot\alpha_3 + \ldots) = 0, \tag{3.38}$$

$$L^{(X)}_2 = -(\frac{1}{2}\alpha_1^2 + \alpha_0\cdot\alpha_2 + \alpha_{-1}\cdot\alpha_3 + \alpha_{-2}\cdot\alpha_4 + \ldots) = 0 \tag{3.39}$$

on $P(\alpha)|k\rangle$. For $P(\alpha) = 1$ we obtain $\hat{k}^2 = k^2/(4\pi T) = -2$ with $\alpha^\mu_0|k\rangle = \hat{k}^\mu|k\rangle$, i.e. we find a scalar, tachyonic state in the string spectrum. On the next level $P(\alpha) = t_\mu\alpha^\mu_{-1}$ the mass shell condition is $k^2 = 0$ and $L^{(X)}_1 = 0$ implies transversality $t_\mu k^\mu = 0$ of the polarization vector t_μ. The norm of this state is proportional to t^2, i.e. it vanishes for a longitudinal polarization $t_\mu \sim k_\mu$. We expect that such a state is Q-exact, and indeed,

$$Q(b_{-1}|\downarrow\rangle) = L^{(X)}_{-1}|\downarrow\rangle + L^{(c)}_{-1}|\downarrow\rangle - b_{-1}Q|\downarrow\rangle \tag{3.40}$$

$$= \hat{k}\cdot\alpha_{-1}|\downarrow\rangle - b_{-1}c_0|\downarrow\rangle - b_{-1}c_0(L_0 - 1)|\downarrow\rangle \tag{3.41}$$

$$= \hat{k}\cdot\alpha_{-1}|\downarrow\rangle + \tfrac{1}{2}\hat{k}^2 b_{-1}c_0|\downarrow\rangle. \tag{3.42}$$

In the case of open strings this is the whole story: We have a massless vector excitation in the target space, whose polarization must be transversal since $Q\alpha^\mu_{-1}|\downarrow\rangle \sim \hat{k}^\mu c_{-1}|\downarrow\rangle$ should vanish, and there is a null (i.e. zero norm) polarization, which is Q-exact. For closed strings we have to include the right-movers and therefore have a polarization tensor $t_{\mu\nu}$ which is transversal in both indices. Now the gauge invariance corresponds to $\delta t_{\mu\nu} = k_\mu v^{(R)}_\nu + k_\nu v^{(L)}_\mu$. The physical interpretation requires the decomposition of the polarization tensor into irreducible representations of the Lorentz group: $t_{\mu\nu}$ has a traceless symmetric part, the graviton, an antisymmetric tensor field $B_{\mu\nu} = t_{\mu\nu} - t_{\nu\mu}$, and a scalar degree of freedom due to the trace, which is called dilaton. The antisymmetric part of the gauge invariance implies that only the field strength $H_{\mu\nu\rho} = \sum \partial_\mu B_{\nu\rho}$ enters physical quantities.

Summarizing our observations, we recover the essential ingredients of QED and of linearized gravity. In addition, we have an infinite tower of gauge symmetries at higher levels which controll the interplay of the infinite set of massive string modes at and above the Planck mass, which, in our picture of string unification, is proportional to $\sqrt{T}$. In order to obtain interaction terms for gravitons and the other target-space fields we need to compute string interactions.

So it seems the 'old covariant approach' to string quantization is sufficient. Ghosts will become important in case of interactions. Before coming to this subject, however, we first continue to the Euclidean domain and set up the machinery of CFT. Since the $SL(2)$ subalgebra of the Virasoro algebra has no anomaly we could, in fact, require $L_n = 0$ for $n \geq -1$. Accordingly, it is useful to work with the $SL(2)$ invariant ghost vacuum $|0\rangle_{gh}$ which is defined by

$$b_n|0\rangle_{gh} = 0 \quad n \geq -1, \qquad\qquad c_n|0\rangle_{gh} = 0 \quad n \geq 2. \tag{3.43}$$

It is related to our previous vacuum by $|\downarrow\rangle = c_1|0\rangle_{gh}$ and $|0\rangle_{gh} = b_{-1}|\downarrow\rangle$. The importance of this vacuum will become clear in the context of CFT on the complex plain.

That $D \leq 26$ is a necessary condition for consistent string quantization can be seen easily by computing the norm of a physical scalar state at the second mass level: We make the ansatz

$$|\phi\rangle = (\alpha_{-1}\cdot\alpha_{-1} + A(\alpha_0\cdot\alpha_{-1})^2 + B\alpha_0\cdot\alpha_{-2})|p\rangle \tag{3.44}$$

Since $L_{n+1} = \frac{1}{n-1}[L_n, L_1]$ it is sufficient to impose $L_0 = L_1 = L_2 = 0$ with $L_0 = L_0^{(X)} - 1$. Straightforward evaluation of the commutators gives $L_0|\phi\rangle = (-\frac{1}{2}\alpha_0^2 + 2 - 1)\alpha_{-1}^2|\phi\rangle$, $L_1|\phi\rangle = 2(1 + A\alpha_0^2 + B)\alpha_0\cdot\alpha_{-1})|p\rangle$ and $L_2|\phi\rangle = (-D - (A - 2B)\alpha_0^2)|p\rangle$, so that we find

$$\frac{\langle\phi|\phi\rangle}{\langle p|p\rangle} = 2D + 4\alpha_0^2 A + 2\alpha_0^4 A^2 - 2\alpha_0^2 B^2 = \tfrac{2}{25}(D-1)(26-D), \qquad B = \tfrac{D-1}{5},\; A = -\tfrac{D+4}{10} \tag{3.45}$$

More generally it can be shown that there are no negative norm states if $a = 1$ and $D = 26$ or if $a \leq 1$ and $D \leq 25$ (see [GR87]). A covariant quantization of string theory below the critical dimension has first been persued successfully by Polyakov [po81]. He found that the conformal anomaly makes the conformal mode ϕ of the metric $g_{mn} = e^{\phi}\eta_{mn}$ dynamical, with an effective Lagrangian (Wess-Zumino term)

$$\tfrac{26-D}{48\pi}\;\left(\tfrac{1}{2}(\partial\phi)^2 + \mu^2 e^{\phi}\right) \tag{3.46}$$

which is positive if $D < 26$. This action has been known for a long time under the name Liouville action and ϕ is thus called Liouville field.

3.4 Strings in background fields

To recover the full content of gravity it seems that we have to study graviton scattering order by order in perturbation theory, which would require the calculation of correlations functions with an arbitrary number of graviton vertex operator insertions. There is, however, an alternative approach that directly gives us the Einstein equations in curved space [fr85, ca$_1$85]. Recall that we can consider the string with an arbitrary target space matric $G_{\mu\nu}(X)$. We computed the equations of motion of the coordinate fields in such a background. It is not so obvious, however, how the dynamics of the background metric arises. As it turns out, this dynamics is fixed by the absence of conformal anomalies.

Here we should be more general: In principle, all massless fields in our theory can form condensates. Hence there should be a consistent movement of strings in curved target spaces with additional backgrounds that correspond to the dilaton and antisymmetric tensor fields. Indeed, if we write down the most general renormalizable action for the coordinate fields we find $S = S_P + S_B + S_\phi + S_\tau$ with

$$\mathcal{L} = \mathcal{L}_P - \tfrac{T}{2}\varepsilon^{mn}\partial_m X^\mu \partial_n X^\nu B_{\mu\nu}(X) + \tfrac{1}{4\pi}\sqrt{-g}\,\phi(X)R^{(2)} + \sqrt{-g}\,\tau(X), \tag{3.47}$$

where $R^{(2)}$ is the curvature scalar on the world sheet. Equations of motion for the coordinate fields read

$$\frac{G^{\alpha\rho}}{\sqrt{-g}\,T}\frac{\delta S}{\delta X^\rho} = \Delta X^\alpha + \partial_m X^\mu \partial_n X^\nu \left(g^{mn}\hat\Gamma_{\mu\nu}{}^\alpha - \tfrac{1}{2}\tfrac{\varepsilon^{mn}}{\sqrt{-g}}H_{\mu\nu}{}^\alpha\right) + O(1/T) \tag{3.48}$$

with the totally antisymmetric 'torsion' $H^{\mu\nu\lambda} = \sum_{\mu\nu\lambda}\partial_\mu B_{\nu\lambda}$, i.e.$H = dB$ with $B = \frac{1}{2}dx^\mu dx^\nu B_{\mu\nu}$ and $H = \frac{1}{3!}dx^\mu dx^\nu dx^\lambda H_{\mu\nu\lambda}$. The contributions of the last 2 terms are suppressed by powers of $1/T$. S_ϕ is conformally invariant only if ϕ is constant and should be considered as contributing only at the quantum level. In 2 dimensions $\sqrt{-g}\,R$ is a total derivative whose integral is proportional to the Euler characteristic of the manifold. Therefore the vacuum expectation value of a constant dilaton field ϕ determines the strength of the string coupling. (S_τ is needed as a counterterm to cancel divergences and plays no further role.)

It can be shown that the resulting quantum theory is conformally invariant to leading order in $1/T$ iff the following 'β-functionals' for the coupling functions G, B and ϕ vanish [fr85, ca$_1$85],

$$0 = R_{\mu\nu} - \tfrac{1}{4}H_{\mu\lambda}{}^\rho H_{\nu\rho}{}^\lambda - 2D_\mu D_\nu\phi \tag{3.49}$$

$$0 = D_\lambda H_{\mu\nu}{}^\lambda - 2H_{\mu\nu}{}^\lambda D_\lambda\phi \tag{3.50}$$

$$0 = 4D_\mu\phi D^\mu\phi - 4D_\mu D^\mu\phi + R + \tfrac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} \tag{3.51}$$

which can be interpreted as equations of motion for the metric, the (field strength of) the antisymmetric tensor field, and the dilaton, respectively. Actually these equations are the Euler Lagrange equations for the (effective) action

$$S_{26} = \int d^{26}X\sqrt{-G}e^{-2\phi}(R - 4D_\mu\phi D^\mu\phi + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}). \tag{3.52}$$

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SKRIPTUM

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Chapter 1

Conformal field theory

So far we developed string theory along canonical lines to a point where we have a gauge fixed free field theory whose dynamical fields $X^\mu(\tau,\sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$ have a Fourier expansion

$$\partial_+ X^\mu = \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{\sqrt{4\pi T}} e^{-in\sigma^+}, \qquad c^+ = \sum_{n=-\infty}^{\infty} c_n e^{-in\sigma^+}, \qquad b_{++} = \frac{i}{2\pi T} \sum_{n=-\infty}^{\infty} b_n e^{-in\sigma^+} \tag{1.1}$$

in case of a flat target space. The canonical commutation relations are

$$[\alpha_m^\mu, \alpha_n^\nu] = n\delta_{m+n}\eta^{\mu\nu}, \qquad [P^\mu, x^\nu] = i\eta^{\mu\nu}, \qquad \{b_m, c_n\} = \delta_{m+n}. \tag{1.2}$$

$p^\mu = \sqrt{4\pi T}\alpha_0^\mu$ is the momentum[1] and x^μ is the center of mass $x = \int_0^{2\pi} d\sigma\, X/(2\pi)$. The physical Hilbert space is defined by the cohomology of the BRST operator

$$Q_+ = \sum_{n=-\infty}^{\infty} :(L_n^{(X)} + \frac{1}{2} L_n^{(c)}) c_{-n}: - a c_0 \tag{1.3}$$

$$= \sum_{n=-\infty}^{\infty} L_n^{(X)}\, c_{-n} - \frac{1}{2} \sum_{n,m=-\infty}^{\infty} (m-n) : c_{-m} c_{-n} b_{m+n} : - a c_0, \tag{1.4}$$

with

$$L_n^{(X)} = -\frac{1}{2} \sum_{m=-\infty}^{\infty} : \alpha_{n-m} \cdot \alpha_m :, \qquad L_n^{(c)} = \sum_{m=-\infty}^{\infty} (n+m) : b_{n-m} c_m : . \tag{1.5}$$

The constant $a = 1$ and the dimension $D = 26$ are fixed by the requirement that $Q^2 = 0$.

Before considering interactions we analytically continue the theory to Euclidean time. The direction of the Wick rotation in the complex time plane is fixed by convergence requirements: The (field) operators[2] at time τ are given by $\mathcal{O}(\tau,\sigma) = e^{i\tau H}\mathcal{O}(0,\sigma)e^{-i\tau H}$. Time ordered correlation functions, therefore, are of the form

$$\langle \mathcal{O}_n(\sigma_n,0) e^{-i(\tau_n-\tau_{n-1})H} \mathcal{O}_{n-1}(\sigma_{n-1},0) \ldots \mathcal{O}_2(\sigma_2,0) e^{-i(\tau_2-\tau_1)H} \mathcal{O}_1(\sigma_1,0) \rangle. \tag{1.6}$$

[1] For open strings the momentum is $P^\mu = p^\mu/2 = \sqrt{\pi T}\alpha_0^\mu$.

[2] Since we have a free field theory, the Heisenberg picture and the interaction picture coincide.

For a positive Hamiltonian this is a convergent expression if the time differences $\tau_i - \tau_{i-1}$ have negative imaginary part. Thus the time evolution should go into the direction of negative imaginary time and we set $\tau = -it$, so that $\sigma^\pm = \tau \pm \sigma = -i(t \pm i\sigma)$.

Instead of considering the complex variables $\xi = i\sigma^+ = t+i\sigma$ and $\overline{\xi} = i\sigma^- = t-i\sigma$ it is useful to map the world sheet cylinder onto the punctured complex plane: The map $\xi \to z = \exp(\xi)$ automatically implements 2π-periodicity in σ and thus is one-to-one. Hence we define

$$z = e^\xi = e^{i\sigma^+}, \qquad \overline{z} = e^{\overline{\xi}} = e^{i\sigma^-}, \qquad \sigma^\pm = \tau \pm \sigma = -i(t \pm i\sigma). \tag{1.7}$$

This transforms the left (right) movers $\partial_\pm X$ to (anti) holomorphic fields on the punctured plane

$$\partial_z X^\mu(z) = \frac{\partial\sigma^+}{\partial z}\,\partial_+ X^\mu(\sigma^+) = \frac{i}{\sqrt{4\pi T}} \sum_n \alpha^\mu_{-n} z^{n-1} \tag{1.8}$$

Time ordering on the world sheet now corresponds to radial ordering on the complex plane, i.e.

$$\mathcal{R}\, A(z)B(w) \;:=\; \theta(|z|-|w|)A(z)B(w) + (-)^{AB}\theta(|w|-|z|)B(w)A(z). \tag{1.9}$$

The vacuum expectation value of the radially ordered product of $\partial X(z)$ with $\partial X(w)$, for example, is

$$\begin{aligned}
\langle \mathcal{R}\,\partial X(z)^\mu \partial X(w)^\nu\rangle &= \tfrac{\theta(|z|-|w|)}{-4\pi T} \sum_{m,n>0} \langle \alpha^\mu_m z^{-m-1} \alpha^\nu_{-n} w^{n-1}\rangle + \tfrac{\theta(|w|-|z|)}{-4\pi T} \sum_{m,n>0} \langle w^{-n-1} z^{m-1} [\alpha^\nu_n, \alpha^\mu_{-m}]\rangle \\
&= \delta^{\mu\nu} \Big(\tfrac{\theta(|z|-|w|)}{-4\pi T} \sum_{n>0} n\, \tfrac{w^{n-1}}{z^{n+1}} + \tfrac{\theta(|w|-|z|)}{-4\pi T} \sum_{n>0} \tfrac{\partial_z}{w} \left(\tfrac{z}{w}\right)^n \Big) \\
&= \tfrac{\delta^{\mu\nu}}{-4\pi T} \left(\theta(|z|-|w|)\partial_w \tfrac{1}{z-w} + \theta(|w|-|z|)\partial_z \tfrac{1}{w-z}\right) = \tfrac{\delta^{\mu\nu}}{4\pi T} \frac{-1}{(z-w)^2}
\end{aligned} \tag{1.10}$$

Here we used that, after continuation to Euclidean space, $[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\delta^{\mu\nu}$ (we should have a Euclidean target space metric to have a Euclidean induced metric on the world sheet). Note that radial ordering and the vanishing of annihilation operators in (vacuum) expectation values work together to make the resulting power series converge for non-equal times. The result is analytic except when z and w coincide.

Integrating with respect to z and w and setting $T = 1/4\pi$ we obtain the propagator up to an integration constant:

$$\langle \mathcal{R}\, X^\mu(z) X^\nu(w)\rangle = -\delta^{\mu\nu} \log(z-w) + \text{const.} \tag{1.11}$$

This agrees with the result for the Green function of a scalar field in 2 dimensions. Indeed, for the Euclidean action[3]

$$S = \frac{1}{4\pi} \int d^2 z\, \partial X(z,\overline{z}) \overline{\partial} X(z,\overline{z}), \qquad \partial := \partial/\partial z, \quad \overline{\partial} := \partial/\partial\overline{z} \tag{1.12}$$

[3] Note that $T/2 = 1/8\pi$, $dxdy = d^2z/2$ and $(\partial_m X)^2 = 4\partial X \overline{\partial} X$ (see below).

the equations of motion are $\frac{1}{2\pi}\partial\overline{\partial}X=0$ and the Green function is $\log|z-w|^2$, because

$$\partial\overline{\partial}\log|z|^2=\pi\delta(x)\delta(y)=2\pi\delta^{(2)}(z),\qquad z=x+iy. \tag{1.13}$$

This can be seen, for example, with the regularization

$$\partial\overline{\partial}\log(|z|^2+\varepsilon)=\partial\,\frac{z}{z\overline{z}+\varepsilon}=\frac{\varepsilon}{(|z|^2+\varepsilon)^2}\to\pi\delta(x)\delta(y)=2\pi\delta^{(2)}(z), \tag{1.14}$$

since, with $|z|=\sqrt{\varepsilon}r$ and for the test function 1, we find $dxdy=rdrd\varphi$ and $\int_0^\infty\frac{r\,dr}{(r^2+1)^2}=\frac{1}{2}$;

$$\begin{array}{lll}\partial:=\partial_z=\frac{1}{2}(\partial_x-i\partial_y), & \partial_x=\partial+\overline{\partial}, & d^2z:=2dxdy=idzd\overline{z},\\ \overline{\partial}:=\partial_{\overline{z}}=\frac{1}{2}(\partial_x+i\partial_y), & \partial_y=i\partial-i\overline{\partial}, & \delta^2(z):=\frac{1}{2}\delta(x)\delta(y)\end{array} \tag{1.15}$$

are the relations between real and complex coordinates [formally, $\delta(x)\delta(y)=-2i\delta(z)\delta(\overline{z})$].

The bad news is that the field $X(z,\overline{z})=X(z)+\overline{X}(\overline{z})$ is not analytic:

$$X^\mu(z)=x^\mu/2+ip^\mu\log z+\sum_{n\neq0}\frac{i}{n}\,\alpha_n^\mu\,z^{-n} \tag{1.16}$$

Considering the usual (field theory) calculation of the propagator as a Fourier transform of $1/k^2$ we observe an IR divergence. In such a situation only physical correlation functions can be expected to be IR finite (see [GR87, p139-149]): Indeed, the good news is that the Hamiltonian and the Virasoro constraints only depend on derivatives of X:

$$T(z)=(\tfrac{i}{z})^2\,T_{++}=\sum_{n=-\infty}^{\infty}L_{-n}z^{n-2}=-\frac{1}{2}:\partial X(z)\partial X(z):+T_{gh}(z) \tag{1.17}$$

(the factor $(i/z)^2$ comes from the transformation of a tensor of rank 2 to the complex plane; recall that $\sigma^+=i\log z$). Also exponentials of X, which will occur in the construction of Vertex operators and in the calculation of scattering amplitudes, are analytic.

Eventually we will be interested in 4-dimensional string theories that can be built by using conformal field theories with $c=22$ together with Minkowski space and the ghost system as building blocks. If $c\neq0$ then T is not a conformal field and has an inhomogeneous transformation law under conformal transformation due to the anomaly term in the Virasoro algebra. For an infinitesimal transformation $z\to z+\xi$ we find

$$\delta_\xi T(z)=\sum\xi_{-m}\,[L_m,T(z)]=\xi(z)\partial T(z)+2\partial\xi(z)T(z)+\tfrac{c}{12}\partial^3\xi(z) \tag{1.18}$$

as can be seen by Laurent expansion in z. For finite transformations $z\to w(z)$ this leads to

$$T(z)=\left(\tfrac{\partial w}{\partial z}\right)^2T(w)+\tfrac{c}{12}S(w,z),\qquad S(w,z):=\frac{\partial w\,\partial^3w-\frac{3}{2}(\partial^2w)^2}{(\partial w)^2}. \tag{1.19}$$

$S(w,z)$, the *Schwartzian derivative* of w w.r.t. z, is the unique object of weight 2 such that $S(w(f(z)),z)=(\partial_zf)^2\,S(w,f)+S(f,z)$ [Gi89]. For the transformation of $T(z)$ from the cylinder to the complex plane this leads to a shift of the zero mode L_0 by $-c/24$.

1.1 $SL_2(\mathbb{C})$ invariance and bc systems

In a similar way we can compute the 2-point function for the analytic parts of the ghost fields

$$c(z)=\sum c_{-n}z^{n+1},\qquad b(z)=\sum b_{-n}z^{n-2}. \tag{1.20}$$

There is, however, an important new issue concerning the correct, or, more precisely, the most useful definition of the vacuum. Let us define a series of ghost 'vacua' $|L\rangle_{bc}$ by the condition that c_n be creation operators for $n\le 1-L$, i.e.

$$\begin{array}{llll} c_n|L\rangle_{bc}\neq 0 & n\le 1-L & c_n|L\rangle_{bc}=0 & n>1-L \\ b_m|L\rangle_{bc}=0 & m\ge L-1 & b_m|L\rangle_{bc}\neq 0 & m<L-1 \end{array} \tag{1.21}$$

The 'up' and 'down' vacua, which are related by $|\downarrow\rangle=b_0|\uparrow\rangle$ and $|\uparrow\rangle=c_0|\downarrow\rangle$, are the special cases $|\downarrow\rangle=|1\rangle_{bc}$ and $|\uparrow\rangle=|2\rangle_{bc}$. For each $|L\rangle_{bc}$ there is a dual state ${}_{bc}\langle L'|$ with ${}_{bc}\langle L'|L\rangle=1$ and ${}_{bc}\langle L'|b_n|L\rangle={}_{bc}\langle L'|c_n|L\rangle_{bc}=0$. Most of these vacua do not have minimal energy, but this does not have a (direct) physical meaning because we are considering the ghost sector.

Now we want to define a scalar product such that $c_n^\dagger=c_{-n}$ and $b_m^\dagger=b_{-m}$. Equivalently, we can fix a mapping from the Fock space to its dual such that $(\mathcal{O}|L\rangle_{bc})^\dagger={}_{bc}\langle L|\mathcal{O}^\dagger$. Therefore we define the *outgoing* (or *Hermitian conjugate*[4]) vacuum ${}_{bc}\langle L|$ by ${}_{bc}\langle L|b_m=0$ iff $b_{-m}|L\rangle_{bc}=0$ and ${}_{bc}\langle L|c_n=0$ iff $c_{-n}|L\rangle_{bc}=0$. This implies that ${}_{bc}\langle L'|$ is proportional to ${}_{bc}\langle 3-L|$. Choosing normalizations, we find

$${}_{bc}\langle L'|L\rangle_{bc}=1,\qquad |L\rangle_{bc}^\dagger={}_{bc}\langle L|={}_{bc}\langle 3-L'|,\qquad |L+1\rangle_{bc}=c_{1-L}|L\rangle_{bc}. \tag{1.22}$$

In particular we have ${}_{bc}\langle 0|c_{-1}c_0c_1|0\rangle_{bc}=1$ and $|\downarrow\rangle=c_1|0\rangle_{bc}$, hence $|0\rangle_{bc}=b_{-1}|\downarrow\rangle$.

To compute the two-point function we insert $b(z)=\sum b_mz^{-m-2}$ and $c(w)=\sum c_{-n}w^{n+1}$ into $\langle b(z)c(w)\rangle_L:={}_{bc}\langle L'|\mathcal{R}b(z)c(w)L\rangle_{bc}$. With $\theta_{z/w}:=\theta(|z|-|w|)=\theta(|z/w|-1)$ we find

$$\langle b(z)c(w)\rangle_L = \theta_{z/w}\sum_{m,n\ge L-1}\langle b_mz^{-m-2}c_{-n}w^{n+1}\rangle_L-\theta_{w/z}\sum_{n,m>1-L}\langle c_nw^{-n+1}b_{-m}z^{m-2}\rangle_L \tag{1.23}$$

$$= \theta_{z/w}\sum_{n\ge L-1}\frac1z\left(\frac wz\right)^{n+1}-\theta_{w/z}\sum_{n>1-L}\frac1z\left(\frac zw\right)^{n-1} \tag{1.24}$$

$$= \theta_{z/w}\,\frac{1}{z-w}\left(\frac wz\right)^{L}-\theta_{w/z}\,\frac wz\frac{1}{w-z}\left(\frac zw\right)^{1-L} = \frac{1}{z-w}\left(\frac wz\right)^{L} \tag{1.25}$$

As it should be, the short distance singularity is independent of L and the correlators only differ by solutions of the homogeneous field equations. But $L=0$ is the only value for which the propagator decays for large time differences $|z|\gg|w|$ *and* for $|w|\gg|z|$. So this is the value which appears to be most appropriate.

[4] More about the definition of *BPZ* [be84] and *Hermitian* conjugation can be found in section 2.2 of [zw93].

In fact, $L_n = \{Q, b_n\} =: L_n^{(X)} : + : L_n^{(c)} : -\delta_{n,0}$ vanishes on $|k=0\rangle \otimes |0\rangle_{bc}$ for $n \geq -1$.[5] Since L_0 and $L_{\pm 1}$ and their right-moving relatives generate an $SL_2(\mathbb{C})$ subalgebra of the Virasoro algebra, $|0\rangle := |0\rangle_{bc}$ is called $SL_2(\mathbb{C})$ invariant vacuum. Our finding that the 'true' ghost vacuum does not have minimal energy appears to tell us about the fact that the ground state of the bosonic string is tachyonic. All these issues will be discussed in more detail when we come to the calculation of correlation functions.

A related argument for $|0\rangle = |0\rangle_{bc}$ being the 'true' vacuum in the ghost sector is the following: Consider a general fermionic first order system with action and energy–momentum tensor given by

$$S = \tfrac{1}{2\pi} \int d^2 z \, (b\bar{\partial} c + \bar{b}\partial\bar{c}), \qquad T_{bc} = -jb\partial c + (1-j)\partial bc \tag{1.26}$$

i.e. the real fields b and c have conformal weights (or dimensions) $h_b = j$ and $h_c = 1-j$. The ghost system of bosonic strings corresponds to the special case $j = 2$. In general such a CFT is called a $b-c$ system with weight j; for $j = 1/2$ we obtain a free Majorana fermion. The Laurent expension of an analytic field ϕ of weight h on the complex plane is given by

$$\phi(z) = \sum_{n=-\infty}^{\infty} \phi_n z^{-n-h}, \qquad \phi_n = \oint \frac{dz}{2\pi i} z^{n+h-1} \phi(z). \tag{1.27}$$

The vacuum state should be a state with 'nothing at the origin'. Therefore $\phi(z)|0\rangle$ should be analytic for small times $\ln|z| \to -\infty$, i.e. the contour integrals

$$\oint \frac{dz}{2\pi i} z^m \phi(z)|0\rangle = \phi_{m-h+1}|0\rangle \tag{1.28}$$

should vanish for $m \geq 0$. We conclude that $\phi_n|0\rangle = 0$ for $n \geq 1-h$. Considering hermitian conjugation $(\phi^\dagger)_n = \phi_{-n}$, we define the outgoing vacuum by $\langle 0|\phi_{-n} = 0$ for $n \geq 1-h$. With this definition of the vacuum the 2-point correlation function is $\langle b(z)c(w)\rangle = \langle c(z)b(w)\rangle = 1/(z-w)$ for general j. For $j = -1$ we recover our $SL_2(\mathbb{C})$ invariant ghost vacuum.

1.2 Operator product expansions

For a conformal field theory with meromorphic quantum fields $\mathcal{O}_i(z)$ and conformal weights h_i we expect that radially ordered operator products can be expanded into a Laurent series

$$\mathcal{O}_i(z)\mathcal{O}_j(w) = \sum_k (z-w)^{h_k - h_i - h_j} \mathcal{C}_{ij}{}^k \mathcal{O}_k(w). \tag{1.29}$$

[5] Recall that $L_n^{(c)} = \sum_m (n+m) : b_{n-m} c_m :$. For $n > 0$ the invariance of $|0\rangle_{bc}$ under L_n is obvious, $L_0 = \ldots - : b_1 c_{-1} : + b_{-1} c_1 + \ldots - \delta_{n,0}$ vanishes since the eigenvalue of $b_{-1}c_1$ is compensated by the last term, and the vanishing of $L_{-1} = \ldots - 2b_0 c_{-1} - b_{-1} c_0 + \ldots$ is again obvious since b_n vanishes for $n \geq -1$.

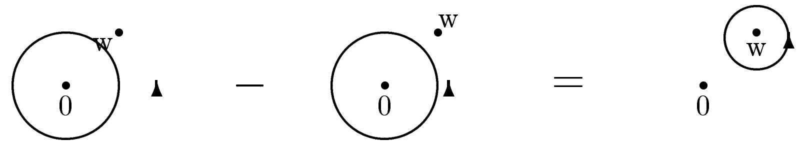

Fig. 6: Commutators and contour integration

Note that in general conformal fields are tensor products of holomorphic and anti-holomorphic components. Then the operator product expansion (OPE) has a more complicated form with operators depending on z and $\overline{z}$ (see below). In any case, as we have seen in the calculation of the 2-point correlation, radial ordering is essential for obtaining well-defined analytic short distance singularities and we should think of the expansions to be inserted into correlations.

With these caveats in mind, we can use these expansions as powerful computational tools: The full mode algebra is, in fact, encoded in the short distance singularities. Deformation of integration contours provides us with a regularization of infinite sums and enables simple and rigorous manipulations.

Consider, for example, a conformal field $\phi(z)$ with weight h, i.e. with the following transformation under $z \to z' = f(z)$:

$$\phi(z) \to \phi'(z) = \left(\frac{\partial z'}{\partial z}\right)^h \phi(z'). \tag{1.30}$$

The conserved quantities $\oint \frac{dz}{2\pi i}\xi(z)T(z)$ generate infinitesimal conformal transformations $z' = z + \xi(z)$ via the equal time commutator with ϕ,

$$\oint \frac{dz}{2\pi i}\xi(z)[T(z), \phi(w)] = \delta_\xi \phi(w) = \xi\partial\phi + h\partial\xi\phi. \tag{1.31}$$

Since lines of equal time correspond to circles around the origin and as integration contours can be deformed as long as no singularities are encountered we can use the following trick to express the commutator in terms of a contour integral (see Fig. 6):

$$\oint \frac{dz}{2\pi i}[\xi(z)T(z), \phi(w)] = \oint_{|z-w|=\varepsilon} \frac{dz}{2\pi i}\xi(z)\mathcal{R}T(z)\phi(w) \tag{1.32}$$

Comparing the last two equations and expanding $\xi(z)$ around w we conclude that the short distance singularity of the OPE $\mathcal{R}T(z)\phi(w)$ must be given by

$$T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w} + \text{regular terms} \tag{1.33}$$

From now on we will usually omit the radial ordering symbol and the symbol $\sim$ will mean equality up to regular terms.

In order to obtain the OPE of $T(z)$ with itself we start with the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n} \qquad (1.34)$$

and compute the transformation of $T(w)$ under the generators

$$T_\xi = \oint \frac{dz}{2\pi i}\xi(z)T(z) = \sum_{n,m}\oint \frac{dz}{2\pi i}\left(\xi_n z^{-n+1}\right)\left(T_m z^{-m-2}\right) = \sum_n \xi_{-n}L_n \qquad (1.35)$$

of conformal transformations. The mode decompositions of derivatives of a conformal field ϕ with weight h read

$$\partial^l\phi = \sum_{N\in\mathbb{Z}} \phi_{-N} z^{N-h-l}(N-h)(N-h-1)\ldots(N-h-l+1). \qquad (1.36)$$

Note that the **number of** consecutive numerical **factors is** the derivative order **l**.
The **sum of** the **exponent** of z **and** of the **index** of ϕ has to be $-l-h_\phi$.
The **largest** numerical **factor is minus** $\mathbf{h_\phi}$ **minus** the **index** of ϕ. Thus we find

$$\begin{aligned}\delta_\xi T(w) &= [T_\xi, T] = \sum_{m,n}\xi_{-n}[L_n, L_m]w^{-m-2} && (1.37)\\ &= \sum_{m,n}\xi_{-n}L_{m+n}w^{-m-2}(n-m) + \frac{c}{12}\sum_n \xi_{-n}w^{n-2}n(n-1)(n+1) && (1.38)\\ &= \xi\partial T + 2\partial\xi T + \frac{c}{12}\partial^3\xi. && (1.39)\end{aligned}$$

For the double sum the last equality is a consequence of $n-m = -(m+n+2)+2(n+1)$; for the sum over n it is obvious from eq. (1.36). Comparing with the above calcuation of $T(z)\phi(w)$ this immediately translates into

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \qquad (1.40)$$

where the relativ factor $2/12 = 1/(3!)$ in the anomalous term comes from the Taylor expansion to 3^{rd} order of $\xi(z)$ around w.

Note that the special form of the structure constants of the operator algebra was essential for being able to write everything in terms of derivatives of meromorphic fields. These structure constants, in turn, are strongly constrained by the Jacobi identities. It can be shown that the Jacobi identities are equivalent to the associativity of the operator algebra. Of course, the Virasoro algebra can be recovered from the singular part of the OPE:

$$\begin{aligned}[L_n, T(w)] &= \left(\oint_{|z|=|w|+\varepsilon} - \oint_{|z|=|w|-\varepsilon}\right)\frac{dz}{2\pi i}z^{n+1}T(z)T(w) && (1.41)\\ &= \oint_{|z-w|=\varepsilon}\frac{dz}{2\pi i}\Big(w+(z-w)\Big)^{n+1}\left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\right) && (1.42)\\ &= (n^3-n)\frac{c}{12}w^{n-2} + 2(n+1)w^nT(w) + w^{n+1}\partial T(w). && (1.43)\end{aligned}$$

Expansion of $T(w)$ around $w = 0$ gives us back the commutators $[L_n, L_m]$. Analogously, for conformal fields of weigth h,

$$[L_n, \phi(z)] = z^n \left(z\partial + (n+1)h\right)\phi(z), \qquad [L_n, \phi_m] = (n(h-1) - m)\,\phi_{n+m} \tag{1.44}$$

are the transformation properties in terms of the Virasoro generators L_n.

The anomalous c-number contributions to eqs. (1.40,1.43), which are called 'Schwinger terms' since Schwinger first observed them in the context of current algebras, modify the conformal transformation of the energy–momentum tensor. Hence T itself is not a 'good' conformal field, i.e. it is not a 'primary' field with conformal transformation (1.30) or, equivalently, with an operator product of the form (1.33) with T. Given the fact that the Schwinger term is field independent, its form follows, up to normalization, from translational invariance and the fact that T has weight 2 under (global) dilatations.

With the technology of 2-dimensional CFT we will avoid manipulations with infinite normal ordered sums or with (operator valued) distributions by encoding everything in OPEs of (operator valued) meromorphic fields. To see explicitly how the analytic continuation avoids singularities consider the equal time anti-commutator

$$\{b(z), c(w)\} = \sum_{m,n} z^{n-2} w^{1-m} \{b_{-n}, c_m\} = \frac{1}{z} \sum_{n\in\mathbf{Z}} \left(\frac{z}{w}\right)^n \tag{1.45}$$

with $z = \exp(t + i\sigma)$ and $w = \exp(t + i\sigma')$. The sum on the r.h.s. of this equation is the Fourier representation of $\delta(\sigma - \sigma')$ (the factor $1/z$ comes from the transformation to the complex plane). With analytic continuation and OPEs we can represent the anti-commutator as

$$\{b(z), c(w)\} = \lim_{\varepsilon\to 0} \left(\mathcal{R}b(z+\varepsilon w)c(w) - \mathcal{R}b(z-\varepsilon w)c(w)\right), \tag{1.46}$$

since the only singularity in the operator product is at $z = w$. Hence, with $w_\varepsilon = w(1+\varepsilon)$ and $\Delta\sigma = \sigma - \sigma'$

$$\{b(z), c(w)\} \approx \frac{1}{z - w_{-\varepsilon}} - \frac{1}{z - w_\varepsilon} \approx \frac{1}{w}\left(\frac{1}{i\Delta\sigma + \varepsilon} - \frac{1}{i\Delta\sigma - \varepsilon}\right) \tag{1.47}$$

In the limit $\varepsilon \to 0$ terms of order $(\Delta\sigma)^2$ in the denominator can be neglected. Using

$$\lim_{\varepsilon\to 0}\left(\frac{1}{x + i\varepsilon} - \frac{1}{x - i\varepsilon}\right) = -2\pi i \delta(x) \tag{1.48}$$

we obtain the result

$$\{b(z), c(w)\} = \frac{1}{iw} \lim_{\varepsilon\to 0}\left(\frac{1}{\Delta\sigma - i\varepsilon} - \frac{1}{\Delta\sigma + i\varepsilon}\right) = \frac{2\pi}{w}\delta(\sigma - \sigma'). \tag{1.49}$$

Differentiating (1.48) it is clear that higher order poles in the OPE are equivalent to derivatives of δ-functions in equal time commutators. One has to be more careful in the calculations, however, since higher order terms in $\Delta\sigma$ may have to be kept in the expansion of the exponential $\exp(\Delta\sigma)$.

1.3 Normal ordered products and Wick theorem

In a concrete CFT the energy–momentum tensor, like other currents and observables, are given in terms of products of elementary field operators. Since such products are usually singular, we need a 'normal ordering' prescription for defining a finite part and we need techniques for computing OPEs of the resulting composite opertors. This will lead us to a generalization of the Wick theorem for general, not necessarily free, conformal fields.

Consider the operator product expansion of two conformal fields $A(z)$ and $B(w)$,

$$A(z)B(w) = \sum_{n=-n_0}^{\infty} [AB]_{-n}(w)(z-w)^n. \tag{1.50}$$

The singular part of this expansion is called the 'contraction' of A and B [FU92, bo93]:

$$\underbrace{A(z)B(w)} = \sum_{n=1}^{n_0} \frac{[AB]_n(w)}{(z-w)^n} \tag{1.51}$$

We assume that the operator algebra is associative and closed in the sense that all coefficients $[AB]_n$ of the OPE, as well as the derivatives of all operators, belong to the algebra.

Now the normal ordered product (NOP) can be defined by subtracting the singularity,

$$[AB](w) \equiv \ :A(w)B(w): \ := \lim_{z\to w}\left(A(z)B(w) - \underbrace{A(z)B(w)}\right) = \oint \frac{dz}{2\pi i}\frac{A(z)B(w)}{z-w}, \tag{1.52}$$

i.e. $[AB] = [AB]_0$. This method of defining a finite part is also called 'point splitting', because we first seperate the positions of the fields A and B by a small distance ε and then take the regular part of the operator product in the limit $\varepsilon \to 0$. In terms of modes this means

$$[AB](w) = \sum_n C_n w^{-n-h_A-h_B}, \qquad C_n = \sum_{n\le -h_A} A_n B_{m-n} + \sum_{n>-h_A} B_{m-n} A_n \tag{1.53}$$

as can be seen by inserting $A(z) = \sum A_n z^{-n-h_A}$ and $B(w) = \sum B_n w^{-n-h_B}$ and deforming the integration contour into the difference between the two circles $|z| = |w| \pm \varepsilon$ in the usual way. Then $1/(z-w)$ has a convergent expansion in w/z and in z/w, respectively. In the first integral the lower limit $0 \le n + h_A$ on n arises form the requirement that the pole orders must not be to high to produce a residue; in the second integral the condition is that we need a pole to get a non-zero contribution, which is the case for $n + h_A \ge 1$.

Note that our definition of the NOP is different from the usual one in QFT, and it is not restricted to free fields. In particular, the NOP (1.53) is not commutative: $[BA] \ne [AB]$. Obviously, the non-commutativity comes from the expansion of the operator product at w

rather than at $\sqrt{zw}$. We can derive a formula that expresses $[BA]_n - [AB]_n$ in terms of derivatives:

$$[BA]_n = \sum_{l=0}^{n_0-n} \frac{(-1)^{n+l}}{l!}\, \partial^l [AB]_{n+l}. \tag{1.54}$$

Proof: Expand $RB(z)A(w) = \sum [BA]_n(w)(z-w)^{-n} = RA(w)B(z) = \sum [AB]_n(z)(w-z)^{-n}$ at w. $\Box$

Note that the NOP is commutative if the contraction of A and B is a **C**-number!

In general the NOP is also not associative. But the non-associativity problem can be administrated nicely because of the **rearrangement lemma**

$$[A[BC]] - [[AB]C] = [B[AC]] - [[BA]C]. \tag{1.55}$$

Proof: We insert the partial fraction decomposition of $\frac{1}{x-z}\frac{1}{y-z}$ into the definition of $[A[BC]]$,

$$[A[BC]](z) = \int\limits_{|x-z|=2\varepsilon} \frac{dx}{2\pi i}\frac{A(x)}{x-z} \int\limits_{|y-z|=\varepsilon} \frac{dy}{2\pi i}\frac{B(y)C(z)}{y-z} = \iint\limits_{|x-z|>|y-z|} \frac{dx\,dy}{(2\pi i)^2}\left(\frac{1}{x-z} - \frac{1}{y-z}\right)\frac{A(x)B(y)C(z)}{y-x}. \tag{1.56}$$

For $[[AB]C]$ we deform the integral of x around y into the difference of two contours around z,

$$[[AB]C](z) = \int\limits_{|y-z|=2\varepsilon} \frac{dy}{2\pi i} \int\limits_{|x-y|=\varepsilon} \frac{dx}{2\pi i}\frac{A(x)B(y)}{x-y}\frac{C(z)}{y-z} = \left(\iint\limits_{|x-z|>|y-z|} \frac{dx\,dy}{(2\pi i)^2} - \iint\limits_{|x-z|<|y-z|} \frac{dx\,dy}{(2\pi i)^2}\right)\frac{A(x)B(y)C(z)}{(x-y)(y-z)}. \tag{1.57}$$

Then the difference $[A[BC]] - [[AB]C]$ is symmeytric under the exchange $A(x) \leftrightarrow B(y)$. $\Box$

To memorize eq.(1.55) observe that the product of a commutator with another operator is associative. The commutator has to be on the l.h.s.; the field on the right cannot be moved away. For more than two fields we fix a default ordering by the recursive definition

$$[A_1 A_2 \ldots A_n] := [A_1[A_2 \ldots A_n]]. \tag{1.58}$$

Note that the contraction operation commutes with differentiation

$$\underbrace{\partial A(z)B(w)} = \partial_z \underbrace{A(z)B(w)}, \qquad \underbrace{A(z)\partial B(w)} = \partial_w \underbrace{A(z)B(w)}, \tag{1.59}$$

so that the first order pole in such contraction vanishes: $[\partial AB]_1 = [A\partial B]_1 = 0$. It is also easy to check that the Leibniz rule is valid for NOPs:

$$\partial[AB] = [\partial AB] + [A\partial B] \tag{1.60}$$

(inserting into the definition of $\partial[AB]$, the term $[\partial AB]$ arisies after partial integration).

We observed that the information of the singular part of the OPE is equivalent to the commutation relations of the Fourier modes of the respective operators:

$$[A_m, B_n] = \left(\oint\limits_{|x|>|y|} \frac{dx}{2\pi i}\oint \frac{dy}{2\pi i} - \oint\limits_{|y|>|x|} \frac{dy}{2\pi i}\oint \frac{dx}{2\pi i}\right) A(x)B(y)\; x^{m+h_A-1}\; y^{n+h_B-1} \tag{1.61}$$

We can, therefore, define an operator algebra of meromorphic fields by stating the contractions of a set of elementary fields, and we should have some criterion for the consisteny of the generated field algebra. The integrand on the r.h.s. of eq. (1.61) has poles only at the origin and at $x = y$. Thus the total integration contour can be deformed into $\oint_0 dy \oint_y dx = -\oint_0 dx \oint_x dy$. We can describe this contour by a formal commutator $[\oint dx, \oint dy]$ with an implicit 'time ordering' of circles, i.e. the integral on the left encloses the origin at a later time. But then the Jacobi identity for this formal commutator is a simple identity for integration contours in a tripple integral. Together with the associativity of the operator algebra[6] this implies that the mode algebra safisfies the Jacobi identity. Explicitly this means that

$$\begin{aligned} \oint_0 \frac{dz}{2\pi i} \oint_z \frac{dy}{2\pi i} \oint_y \frac{dx}{2\pi i} \; \underbrace{\underbrace{A(x)B(y)}\,C(z)} \; f(x,y,z) \quad &+ \\ \oint_0 \frac{dx}{2\pi i} \oint_x \frac{dz}{2\pi i} \oint_z \frac{dy}{2\pi i} \; \underbrace{\underbrace{B(y)C(z)}\,A(x)} \; f(x,y,z) \quad &+ \\ \oint_0 \frac{dy}{2\pi i} \oint_y \frac{dx}{2\pi i} \oint_x \frac{dz}{2\pi i} \; \underbrace{\underbrace{C(z)A(x)}\,B(y)} \; f(x,y,z) \quad &= \quad 0 \end{aligned} \tag{1.62}$$

for all functions $f(x, y, z)$ that are analytic on the punctured complex plane $\mathbf{C}^* = \mathbf{C} - \{0\}$. This equation is called **associativity of the operator product algebra** (the contraction operations can be omitted without changing the integrals). We thus found the following result: Associativity of the operator algebra implies the Jacobi identity for the mode algebra, which, in turn, is equivalent to eq. (1.62). So this is at least a necessary condition (see, for example, [bo91]). It is straightforward to check (1.62) for a given set of contractions: The first two integrals are evaluated by Taylor–expanding f to the appropriate order (we may assume that $f(x, y, z) = f(x)g(y)h(z)$). Then the integrand for the final integral must be a total derivative.

The important rule for computing OPEs of composite operators is the **Wick theorem**:

$$\underbrace{A(z)[BC](w)} = \oint_w \frac{dv}{2\pi i} \frac{\underbrace{A(z)B(v)}\,C(w)}{v-w} \; + \; [B(w)\,\underbrace{A(z)C(w)}] \tag{1.63}$$

Proof: The singularities of the operator product $A(z)B(v)C(w)$ as a function of z near v and w are given by the contractions of $A(z)$ with $B(v)$ and $C(w)$. Integrating $dv/(v - w)$ around w,

$$\underbrace{A(z)[BC](w)} = \oint_w \frac{dv}{2\pi i} \frac{\underbrace{A(z)B(v)}\,C(w) + B(v)\,\underbrace{A(z)C(w)}}{v-w}, \tag{1.64}$$

we obtain the Wick theorem. □

The last term in (1.64) can be simplified to give the normal product of B with the contraction of A and C, but the integral with the contraction of A and B has to be evaluated carefully: If

[6] Note that the identity $R\,(A(x)B(y))\;C(z) = R\,(B(y)C(z))\;A(x)$ involves some analytic continuation.

$\underbrace{A(z)B(v)}$ and $C(w)$ have a short distance singularity then terms in the expansion of $1/(z-v)^n$ around w,

$$\frac{1}{(z-v)^n} = \frac{1}{(z-w)^n} + \binom{n}{1}\frac{(v-w)}{(z-w)^{n+1}} + \binom{n+1}{2}\frac{(v-w)^2}{(z-w)^{n+2}} + \ldots \tag{1.65}$$

can combine with poles $1/(v-w)^m$ to produce a residue in the v integration.

In terms of the operator product coefficients the Wick theorem thus reads

$$[A[BC]]q = [B[AC]_q] + \sum_{l=0}^{q-1} \tbinom{q-1}{l}[[AB]_{q-l}C]_l \qquad q>0 \tag{1.66}$$

(we always omit the obvious sign factors in case of fermions). For $q=0$ the rearrangement lemma tells us that there is an additional normal ordered commutator on the r.h.s. of this expression: $[A[BC]] = [B[AC]] + [([AB]-[BA])C]$. If the contraction $\underbrace{A(z)B(w)}$ is a C-number function, i.e. if all $[AB]_q$ are proportional to the identity for $q>0$, so that only $l=0$ contributes in the above sum, then the Wick theorem reduces to the usual expression for free fields: $\underbrace{A[BC]} = [\underbrace{AB}C] + [B\underbrace{AC}]$. In particular, by iteration of this equation,

$$\underbrace{A(z)B^n(w)} = n\,\underbrace{A(z)B(w)}\,B^{n-1}(w), \qquad \underbrace{A(z)e^{B(w)}} = \underbrace{A(z)B(w)}\,e^{B(w)}. \tag{1.67}$$

whenever $\underbrace{A(z)B(w)} \in \mathbf{C}$.

The situation is more complicated if there is a composite operator on the left. As an example we compute the central charge of a free boson. Since X itself is not a conformal field (the 2-point function is a logarithm) we introduce $J = \partial X$, i.e. $\underbrace{J(z)J(w)} = -1/(z-w)^2$ and $T = -J^2/2$. It is easy to check that J is a conformal field of weight 1,

$$\underbrace{T(z)J(w)} = \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}. \tag{1.68}$$

The contraction of T with itself is

$$\underbrace{T(z)T(w)} = -\frac{1}{2}\oint_w \frac{dv}{2\pi i}\left(\frac{J(v)}{(z-v)^2} + \frac{\partial J(v)}{z-v}\right)\frac{J(w)}{v-w} - \frac{1}{2}[J(w)\,\underbrace{T(z)J(w)}] \tag{1.69}$$

$$\begin{aligned} &= -\frac{1}{2}\oint_w \frac{dv}{2\pi i}\left(\frac{-1}{(v-w)^3} + \frac{J^2(w)}{v-w}\right)\left(\frac{1}{(z-w)^2} + 2\frac{v-w}{(z-w)^3} + 3\frac{(v-w)^2}{(z-w)^4}\right) \\ &\quad -\frac{1}{2}\oint_w \frac{dv}{2\pi i}\left(\frac{2}{(v-w)^4} + \frac{[J\partial J](w)}{v-w}\right)\left(\frac{1}{z-w} + \ldots + \frac{(v-w)^3}{(z-w)^4}\right) \\ &\quad -\frac{1}{2}\left(\frac{J^2(w)}{(z-w)^2} + \frac{[J\partial J](w)}{z-w}\right) \end{aligned} \tag{1.70}$$

$$= \frac{3/2 - 2/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{1.71}$$

so the contribution of a free boson to the central charge is $c = 1$. Note that the short distance singularity of $J(v)J(w)$, together with the expansion of $1/(z-v)$ around w, produces the central term. With the Wick theorem of QFT this term would come from the double contractions.

To compute the critical dimension of the bosonic string, we still need the central charge of the ghost system. In fact, with almost no extra effort, we can also obtain the critical dimension of the fermionic string. To this end we consider a general first order system with energy–momentum tensor

$$T_{bc} = (1-j)[\partial bc] - j[b\partial c], \qquad \underbrace{b(z)c(w)} = \varepsilon\, \underbrace{c(z)b(w)} = \frac{\varepsilon}{z-w} \tag{1.72}$$

with $\varepsilon = 1$ for fermions and $\varepsilon = -1$ for bosons (i.e. for a so-called $\beta - \gamma$ system). Then

$$\underbrace{T_{bc}(z)b(w)} = \frac{jb(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w}, \qquad \underbrace{T_{bc}(z)c(w)} = \frac{(1-j)c(w)}{(z-w)^2} + \frac{\partial c(w)}{z-w}. \tag{1.73}$$

Trusting that the OPE of T_{bc} with itself is of the correct form for an energy–momentum tensor, we only need to compute the central term. So we use (1.66) to directly evaluate the 4^{th} order pole term. Only the last term on the r.h.s. of that equation can contribute, hence

$$\frac{c}{2} = [T_{bc}T_{bc}]_4 = (1-j)[T_{bc}[\partial bc]]_4 - j[T_{bc}[b\partial c]]_4 \tag{1.74}$$

$$= (1-j)\sum_{l=1}^{3} \binom{3}{l}[[T_{bc}\partial b]_{4-l}c]_l - j\sum_{l=2}^{3} \binom{3}{l}[[T_{bc}b]_{4-l}\partial c]_l \tag{1.75}$$

$$= 6(1-j)j\ [bc]_1 + 3(1-j^2)\ [\partial bc]_2 + (1-j)\ [\partial^2 bc]_3 - 3j^2\ [b\partial c]_2 - j\ [\partial b\partial c]_3 \tag{1.76}$$

$$= 6(1-j)j\ \varepsilon + 3(1-j^2)\ (-\varepsilon) + (1-j)\ 2\varepsilon - 3j^2\ \varepsilon - j\ (-2\varepsilon) \tag{1.77}$$

$$= \varepsilon(6j(1-j) - 1), \tag{1.78}$$

where we used

$$\underbrace{T_{bc}(z)\partial b(w)} = \frac{2jb(w)}{(z-w)^3} + \frac{(j+1)\partial b(w)}{(z-w)^2} + \frac{\partial^2 b(w)}{z-w}, \qquad \underbrace{b(z)c(w)} = \frac{\varepsilon}{z-w}. \tag{1.79}$$

It should not be too surprising that the dependence on the statistics of b and c is only through an overall sign. The central charge of the ghost system is $c = 12j(1-j) - 2 = -26$, so the total central charge for the bosonic string is $c = D - 26$, which vanishes in the 'critical dimension' $D = 26$.

Anticipating the particle content of 2-dimensional supergravity we can now also compute the critical dimension for the fermionic string: Supersymmetry gives a fermionc partner to all bosons, so we get D Majorana fermions and one gravitino. A $b-c$ system with $j = 1-j = 1/2$ corresponds to two real fermions, so the D superpartners of the real coordinate functions contribute $c = D/2$. Superconformal gauge fixing eliminates the gravitino, but generates a bosonic ghost system: The bosonic partner β of the fermionic lagrange multiplyer has spin

```
Bosonic[dX];
OPE[dX,dX]=MakeOPE[{−One,0}];
T=−NO[dX,dX]/2;
OPESimplify[OPE[T,T],Together]
```

```
Fermionic[b,c];
OPE[b,c]=MakeOPE[{One}];
T=(1−j) NO[b',c]−j NO[b,c'];
Factor[ OPEPole[4][T,T] ]
```

Table I: Calculation of the central charge for the bosonic string with OPEdefs.m

$j = 3/2$ and the parameter field of the local supersymmetry transformations, which turns into the bosonic ghost γ, has spin $-1/2 = 1 - j$. Thus the (bosonic) superconformal ghosts are a $\beta - \gamma$ system with spin $j = 3/2$. Altogether we find $c = D\,3/2 - 26 + 11 = (D - 10)\,3/2$, so the critical dimension is 10.

With our machinary it is straightforward to compute any OPE of composite operators in terms of the contractions of a set of elementary fields. But in practice this may be very tedious and it is easy to make mistakes. Fortunately there is the Mathematica package 'OPE-defs.m' , written by K. Thielemans [th91], which does the job for us. The above calculation of the central charges for free bosons and for a $b - c$ system, for example, can be done on a computer by loading the package into a Mathematica session (with "<<OPEdefs.m") and by typing the commands that are listed in table I. To compute the central charge for a $\beta - \gamma$ system, we just need to replace *Fermionic[b,c]* by *Bosonic[B,C]* and *OPE[b,c]=MakeOPE[One]* by *OPE[B,C]=MakeOPE[-One]*.

A conformal field $J(z)$ with weight $h_J = 1$ provides a charge $Q_J = J_0 = \oint \frac{dz}{2\pi i} J(z)$ that commutes with T, and hence with all Virasoro generators L_n, because

$$\underbrace{T(z)J(w)} = \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} = \partial_w \frac{J(w)}{z-w} \tag{1.80}$$

is a total derivative. To each $b - c$ or $\beta - \gamma$ system we have associated, in addition to the energy–momentum tensor, the classically conserved number current $J_c(z) = -[bc](z)$ of weight 1. We have seen in the case of the ghost system that expectation values of operators sandwiched between $SL(2, \mathbf{C})$ vacua vanish unless the ghost number of the operator is 3. For general spin, this quantum mechanical violation of 'ghost' number is by an amount $Q = \varepsilon(1 - 2j)$, the so-called 'background charge'. It also shows up in an anomalous term in the OPE

$$T_{bc}(z)J_c(w) \sim \frac{Q}{(z-w)^3} + \frac{J(z)}{(z-w)^2}. \tag{1.81}$$

The remaining OPEs are

$$J_c(z)c(w) \sim \frac{c(w)}{z-w}, \qquad J_c(z)b(w) \sim \frac{-b(w)}{z-w}, \qquad J_c(z)J_c(w) \sim \frac{\varepsilon}{(z-w)^2}. \tag{1.82}$$

In terms of Q, the contribution to the conformal anomaly is $c = \varepsilon(1 - 3Q^2)$. Note that Q vanishes for $j = 1/2$, so that fermion number is conserved.

Returning to the bosonic string, i.e. $j = 2$ and $D = 26$, we still need to discuss the BRST operator $Q_{BRST} = \oint j_Q$ and the OPEs of the BRSTcurrent j_Q. Naively, we would take $j_Q = cT_x + \frac{1}{2}cT_{bc}$, whose OPE with $T = T_x + T_{bc}$ is

$$T(z)[cT_x + \tfrac{1}{2}cT_{bc}](w) \sim \frac{9c(w)}{(z-w)^4} + \frac{3\partial c(w)}{(z-w)^3} + \frac{[cT_x + \frac{1}{2}cT_{bc}](z)}{(z-w)^2}, \tag{1.83}$$

so that this expression is not a conformal field. The non-covariant terms, however, are the same as the ones in the OPE of $T(z)$ with $-\frac{3}{2}\partial^2 c(w)$. A Noether current is only defined up to total derivatives, so we can choose a BRST-current

$$j_Q := cT_x + \tfrac{1}{2}cT_{bc} + \tfrac{3}{2}\partial^2 c = -\tfrac{1}{2}c\,\partial X_\mu\,\partial X^\mu + b\,c\,\partial c + \tfrac{3}{2}\partial^2 c, \tag{1.84}$$

which is a conformal field with weight 1 (here all NOPs are associative and commutative).

The OPEs of j_Q with J_c and with b are

$$\underbrace{j_Q(z)J_c(w)} = \partial_w \frac{-2c(w)}{(z-w)^2} + \frac{j_Q(w)}{z-w}, \qquad \underbrace{j_Q(z)b(w)} = \frac{3}{(z-w)^3} + \frac{J_c(w)}{(z-w)^2} + \frac{T(w)}{z-w}. \tag{1.85}$$

The OPE of j_Q with itself is a total derivative in 26 dimensions:

$$\underbrace{j_Q(z)j_Q(w)} = \partial_w \frac{2[\partial c\ c](w)}{(z-w)^2} \tag{1.86}$$

Note that $(\partial c)^2 = 0$, which follows from the identity[7]

$$[FF](w) = -\tfrac{1}{2}\sum_{l>0} \frac{(-)^l}{l!}\partial^l [FF]_l \qquad (F \text{ fermionic}) \tag{1.87}$$

for fermionic operators F. This identity, in turn, is a consequence of (1.54)).

1.4 Vertex operators

So far we only considered derivatives of the target space coordinates, but there is another way to make conformal fields out of $X^\mu(z)$. In order to create a 'coherent' string state with momentum k we need to apply a 'vertex operator'

$$V_k(z) = :e^{ikX(z)}: \tag{1.88}$$

to the vacuum. Since the contractions among coordinates X^μ have logarithmic singularities in the complex plane, we need to define this normal ordered expression by the usual normal

[7] According to ref. [th91], OPEdefs.m uses the rules (1.54), (1.55), (1.60), (1.66) and (1.87).

ordering (fortunately, this is consistent with (1.53) since ∂X has weight 1, so that α_n^μ are annihilators for $n > 0$). Also, its elementary contractions with conformal fields have to be computed using the conventional Wick theorem for free fields.

Since $\langle\partial X^\mu(z)X^\nu(w)\rangle = -\delta^{\mu\nu}/(z-w)$, the OPE of V_k with $i\partial X^\mu$ and the derivative of a vertex operator are

$$i\partial X^\mu(z)V_k(w) \sim \frac{k^\mu}{z-w}V_k(w), \qquad \partial V_k(z) = ik_\mu :\partial X^\mu V_k:(z) = k_\mu[P^\mu V_k](z). \tag{1.89}$$

The vector of charges $P^\mu = i\int\frac{dz}{2\pi i}\partial X^\mu$, which is, of course, the string momentum, has eigenvalues $[P^\mu, V_k(z)] = kV_k(z)$ and is conserved in correlation functions. To compute the OPEs among Vertex operators we insert the series for the exponential,

$$\begin{aligned}\underbrace{:e^{A(z)}: \;\; :e^{B(w)}:} &= \sum_{m,n\geq 0} \underbrace{:A^m(z): \;\; :B^n(w):}\;\tfrac{1}{m!n!} \\ &\sim \sum_{m,n,l} \tfrac{l!}{m!n!}\tbinom{m}{l}\tbinom{n}{l}\big(\underbrace{A(z)B(w)}\big)^l :A^{m-l}(z)B^{n-l}(w): \;\; = e^{\overbrace{A(z)B(w)}} :e^{A(z)}e^{B(w)}:,\end{aligned} \tag{1.90}$$

hence

$$\underbrace{V_k(z)V_q(w)} \sim (z-w)^{kq} :e^{ikX(z)}e^{iqX(w)}: \tag{1.91}$$

Setting $k+q=0$ we see that V_k should be a conformal field with dimension $h(V_k) = k^2/2$. Indeed, the OPE of the vertex operator with the energy–momentum tensor is

$$\underbrace{T(z)V_k(w)} = \frac{k^2/2}{(z-w)^2}V_k(w) + \frac{\partial V_k(w)}{z-w}. \tag{1.92}$$

To rewrite (1.91) with our normal ordered products we would need to expand $V_k(z)$ around w to order $O(z-w)^{-kq}$ within the normal ordering symbol, which then could be replaced by our NOP.[8]

1.5 Ward identities and conformal bootstrap

In the context of Euclidean field theories Greens functions $\langle R\mathcal{O}_1\ldots\mathcal{O}_n\rangle$ are usually called correlation functions. So far we were mainly interested in 'conformal fields' ϕ_i which transform like tensors with a certain conformal weight under conformal coordinate transformations. These fields are called *primary fields*. There are also other field operators, like $T(z)$, derivatives, or the non-leading coefficients in OPEs.

[8] The problem in finding a version of (1.91) that is correct for conformal fields with $\mathbf{C}-$number contractions is that the exponential of a pole is an essential singularity. At best, we would have to keep infinitely many terms in the expansion of $A(z)$ around w.

The fields that can be obtained as multiple commutators of a primary field with Virasoro generators L_{-n} with $n > 0$ are called *descendents* [be84]. A primary field ϕ together with its descendents is called a conformal family $[\phi]$. Since $[L_n, \phi] = 0$ for $n > 0$ and $[L_0, \phi] = h\phi$, it is easy to see that $[\phi]$ is a 'highest weight' representation (or module) of the Virasoro algebra. In an axiomatic approach [be84, mo89][9] it is assumed that all field operators are linear combinations of members of conformal families, i.e. the Hilbert space is a direct sum $\mathcal{H} = \oplus_{h,\overline{h}} \left(V(h,c) \otimes \overline{V}(\overline{h},\overline{c})\right)$ of products $V \otimes \overline{V}$ of conformal families $V(h_i, c) = [\phi_i]$. If this sum is finite, then the CFT is called rational.

It can be shown that a unitary CFT is rational (with respect to the Virasoro algebra) iff[10] $c < 1$ [ca86] (if this is the case then $c = 1 - \frac{6}{n(n+1)}$ for some integer $n \geq 2$ [be84, fr84]). It is possible, however, to generalize the definition of rational theories in the following way. The *chiral* fields of a CFT, i.e. the fields with $\overline{h} = 0$, form a subalgebra of the operator algebra which is called the (left) 'chiral algebra' $\mathcal{A}_L$. If this algebra is larger then the Virasoro algebra, then all fields can be grouped into representations of that larger algebra. We call a CFT rational if the number of such representations is finite. Important examples of such a situation are (extended) supersymmetries and 'Kac–Moody' (current) algebras. If the extension is by fields of higher conformal weight $h \geq 3$ then such an algebra is called a W algebra [bo93] (such algebras are in general non-linear, i.e. they are not inifinite-dimensional Lie algebras). In this way infinitely many fields that are Virasory-primary can become descendents, i.e. they can be obtained as commutators with generators of $\mathcal{A}_L$. Rational CFTs, therefore, can exist for arbitrarily large values of c, but for a given chiral algebra there will always be a maximal value bejond which no rational theories exist.

The correlation functions of primary fields with one energy-momentum tensor satisfy the *conformal Ward identity*

$$\langle T(z)\phi_1(w_1)\ldots\phi_n(w_n)\rangle = \sum_i \left(\frac{h}{(z-w_i)^2} + \frac{\partial_{w_i}}{z-w_i}\right)\langle\phi_1(w_1)\ldots\phi_n(w_n)\rangle. \tag{1.93}$$

The OPE of T with ϕ_i imply that the singularieties of the correlations as a function of z agree on both sides. But both sides are holomorphic functions and the l.h.s. should decay for $z \to \infty$. This implies (1.93), since a holomorphic function that vanishes at infinity must vanish on the complex plane. Considering various contour integrals of the Ward identity multiplied with meromorphic functions of z we can, therefore, compute correlation functions of descendent

[9] The following set of axioms is used in [mo89]:
1. There is a unique $SL_2(R) \times SL_2(R)$ invariant vacuum with $h = \overline{h} = 0$,
2. For each vector $\alpha \in \mathcal{H}$ there is an operator ϕ_α and its (charge) conjugate,
3. For 'highest weight states' $\alpha = i$ we have $[L_n, \phi_i(z,\overline{z})] = (z^{n+1}\partial_z + h_i(n+1)z^n)\phi_i$,
4. $\langle 0|\mathcal{O}_1(z_1,\overline{z}_1)\ldots\mathcal{O}_n(z_n,\overline{z}_n)|0\rangle$ exist for $|z_i| > |z_{i+1}|$ and have an analytic continuation to $\mathbf{C}^n$ for $z_i \neq z_j$.
5. One loop correlation functions exist and are modular invariant.

[10] Unitarity implies that $c/2 = \langle 0|[L_2, L_{-2}]|0\rangle = \langle 0|L_2 L_{-2}|0\rangle \geq 0$.

fields, once the correlation functions of the primaries are known (it should be obvious how this can be extended to higher descendents).

In the literature the conformal Ward identity is often derived directly from the path integral. The OP singularities of T with primary fields are then obtained as a consequence of (1.93). In fact, we do not really have field operators in that approach, but the OPEs still can be understood as expansions that are correct when inserted into correlation functions.

The generators L_n with $|n| \le 1$ span an anomaly free subalgebra of the Virasoro algebra. The corresponding group of finite transformations

$$z \to \frac{az+b}{cz+d}, \tag{1.94}$$

is isomorphic to (projective) $SL(\mathbf{C})$. (1.94) are called Möbius transformations, or global conformal transformations, since they are the only non-singular conformal maps of the compactified complex plane onto itself.

Fields that transform under global conformal transformations with a certain exponent of the functional determinant are called quasi-primary. This notion is not restricted to two dimensions, since translation, rotations, dilatations and special conformal transformations are conformal symmetries of flat space in arbitrary dimensions. Since we use the $SL(2,\mathbf{C})$ invariant vacuum, correlation functions of quasi-primary fields must also be $SL(2,\mathbf{C})$ invariant. This implies that the 2-point functions are of the form

$$\langle\phi_i(z)\phi_j(w)\rangle = \frac{C_{ij}}{(z-w)^{h_i+h_j}}, \tag{1.95}$$

because translations and rotations imply that the l.h.s. only depends on $z-w$ and dilatations fix the exponent. Invariance under inversion $z \to -1/z$ further implies that $C_{ij}=0$ if $h_i \ne h_j$ because this is necessary for $(\frac{\partial z'}{\partial z})^{h_i}(\frac{\partial w'}{\partial w})^{h_j} = (\frac{z'-w'}{z-w})^{h_i+h_j}$.

Using the same arguments for 3-point correlations we find

$$\langle\phi_i(z_i)\phi_j(z_j)\phi_k(z_k)\rangle = \sum_{abc}\frac{C_{abc}}{r_{ij}^a r_{jk}^b r_{ik}^c}, \qquad r_{ij}=z_i-z_j \tag{1.96}$$

with $a+b+c=h_i+h_j+h_k$. Invariance under inversion further implies $a=h_1+h_2-h_3$, $b=h_2+h_3-h_1$ and $c=h_1+h_3-h_2$, i.e.

$$\langle\phi_i(z_i)\phi_j(z_j)\phi_k(z_k)\rangle = \frac{C_{ijk}}{r_{ij}^{h_1+h_2-h_3} r_{jk}^{h_2+h_3-h_1} r_{ik}^{h_1+h_3-h_2}}. \tag{1.97}$$

With similar arguments on can show [Gi89] that higher correlations only depend on cross ratios $(r_{ij}r_{kl})/(r_{ik}r_{jl})$. For 4-point functions there are 2 independent cross ratios and we have a dependence on an arbitrary function F,

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = F(\frac{r_{12}r_{34}}{r_{13}r_{24}},\frac{r_{12}r_{34}}{r_{23}r_{14}})\prod_{i<j}(r_{ij})^{h/3-h_i-h_j} \tag{1.98}$$

with $h = \sum h_i$. It is no surprise that we need four points to get a non-trivial coordinate dependence, since 3 points can always be fixed to, say, $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$ by Möbius transformations.

1.6 Scattering amplitudes

For the description of string scattering we use the conformal invariance to map a 'tree level' world sheet to the punctured complex plane. Then the asymptotic states are generated by operator insertions at the punctures. The sum over all surfaces in the path integral now corresponds to a sum over all conformally inequivalent metrics and over all positions of the punctures on the world sheet. The classes of metrics correspond to inequivalent Riemann surfaces. Hence, at genus 0 there is only the integral over the positions of the punctures.

As our next step we need to find out which operator insertions correspond to physical string states. Gauge independence is guaranteed if we only consider correlations of BRST invariant operators $\phi(z)$:

$$[Q, \phi(w)] = \oint_w \frac{dz}{2\pi i} j_{BRST}(z)\phi(w) = \text{total derivative.} \tag{1.99}$$

Since we are interested in integrated correlations we may allow for a total derivative on the r.h.s. of this equation, so the first order pole in the OP of j_{BRST} with ϕ should vanish or be a total derivative. Consider operators without ghost excitations. Then

$$\oint \frac{dz}{2\pi i} j_{BRST}(z)\phi(w) = \oint \frac{dz}{2\pi i} c(z)T^{(X)}(z)\phi(w) = \oint \frac{dz}{2\pi i} c(z) \left(\frac{h_\phi \phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w} \right) \tag{1.100}$$

$$= h_\phi (\partial c)\phi(w) + c\partial\phi(w) \tag{1.101}$$

This is a total derivative iff $h_\phi = 1$. Of course, the same considerations apply to the right-moving components. Hence, in particular, the 'tachyon vertex operator' $: \exp(ik_\mu X^\mu(z, \overline{z}) :$ is physical if $k^2/2 = -1$. This is called the on-shell condition for obvious reasons.

Alternatively, we can use the correspondence of operators and asymptotic 'incoming' states by the relation $\psi|0\rangle = |\psi\rangle$. Let us consider states of the form

$$|\psi\rangle = |\phi\rangle^{(X)} \otimes |\downarrow\rangle \tag{1.102}$$

Then BRST invariance $Q|\psi\rangle = 0$ is equivalent to the physical state condition

$$(L_0^{(X)} - 1)|\phi\rangle = 0, \qquad L_n^{(X)}|\phi\rangle = 0 \quad n > 0. \tag{1.103}$$

Such a state is generated from the vacuum by a vertex operator $\psi(z) = \phi(z)c(z)$, which commutes with Q iff $h_\phi = 1$.

The two possible forms of vertex operators with ghost number 0 and 1 are related by the following contour integration

$$V_k(w) = \oint \frac{dz}{2\pi i} b(z)c(w)V_k(w). \tag{1.104}$$

In order to obtain a non-vanishing result for an n-point function we need to have a total ghost number insertion of 3. Tree level amplitudes are therefore obtained by inserting 3 BRST-invariant vertex operators $c(z)\phi(z)$ and BRST-invariant integrals $\int dz\phi(z)$ for the remaining external legs. Since the amplitude is invariant under global conformal transformations we are free to fix 3 positions of the insertion to some arbitrary values. For the simplest example of tachyon scattering, with $\phi = V_k$ and $k^2 = -2$, the resulting n-point function

$$\langle \prod_{i=1}^{3} c(z_i)V_{k_i}(z_i) \prod_{i=4}^{n} V_{k_i}(z_i)\rangle \tag{1.105}$$

is proportional to

$$\prod_{i<j\le 3} (z_i - z_j) \prod_{i<j} (z_i - z_j)^{k_i k_j} \delta(\sum k_i). \tag{1.106}$$

The first factor comes from the ghost insertions: Since $\langle c_{-1}c_0c_1\rangle = 1$ is the basic non-vanishing correlation, we have

$$\langle 0|c(z_1)c(z_2)c(z_3)|0\rangle = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{vmatrix} = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3). \tag{1.107}$$

($1, z$ and z^2 are just the zero modes of c, i.e. normalizable solution to the ghost equations of motion, the global conformal Killing vector fields; see below). The second factor is the product of all short distance singularities of the products of the vertex operators $V_k(z)$ [see (1.91)]. Since the correlation is a meromorphic function of all z_i, the first two factors contain all z dependence of the correlation. Since momentum is conserved, we get an additional δ-function for the sum of all momenta.

The factor coming from the ghosts can also be understood as contribution from the gauge fixing functional determinant to the measure in the path integral: Consider the infinitesimal form of the $SL(2,\mathbf{C})$ transformations $z \to z' = \frac{az+b}{cz+d}$ with $ad - bc = 1$ paramtetrized as $a = 1 + \alpha/2$, $b = \beta$, $c = \gamma$, $d = 1 - \alpha/2$, i.e. $z' \sim z + \beta + \alpha z - \gamma z^2$.

$$\left|\frac{\partial(z_i, z_j, z_k)}{\partial(\alpha, \beta, \gamma)}\right| = (z_i - z_j)(z_i - z_k)(z_j - z_k) \tag{1.108}$$

is the functinonal derterminant for fixing the positions z_i, z_j and z_k.

Fixing $z_1 = 0$, $z_2 = 1$, $z_4 = \infty$ and using the on-shell conditions we thus obtain the Virasoro–Shapiro amplitude

$$A = \int d^2z |1-z|^{2p_2p_3} |z|^{2p_3p_1} \tag{1.109}$$

for scattering of two tachyons.

For open strings a similar calculation leads to the Veneziano amplitude

$$A=\int_0^1 dz(1-z)^{p_2p_3}z^{p_3p_1}. \tag{1.110}$$

Here the vertex for string absorption/emission has to be inserted at the boundary of the upper half plane and the total amplitude is a sum over the different zyclic orderings of the insertions, i.e. we have to sum the above amplitude over s, t and u channel (for closed strings the amplitude is 'dual' without this sum because points in the interiour of the world sheet can be moved around one another. Inserting the OPE of the vertex operators we can 'factorize' the amplitude in one of the channels. In this way we get a sum over pole terms, corresponding to the poles of physical particles. This is another way to obtain the mass spectrum of string states.

1.7 Ghost number anomaly and topology

The ghost number violation that forces us to insert ghosts into physical correlation functions can also be understood directly from a path integral consideration: Recall that the total gauge fixed action (with the lagrange multipliers, the Weyl ghost and the trace part of the anti-ghost integrated out, is

$$\mathcal{L}=\mathcal{L}_P+\tfrac{T}{2}\int\sqrt{-g}\,b^{mn}(Pc)_{mn} \tag{1.111}$$

where the operator P with $(Pc)_{mn}=D_mc_n+D_nc_m-g_{mn}Dc$ maps vector fields into traceless symmetric tensor fields. The (path) integral over a fermionic variable vanishes if the integrand does not depend on that variable. Therefore we need to insert an extra ghost for any zero mode of P and an extra anti-ghost for any zero mode of $P^\dagger$. This implies a total ghost number violation of the number of zero modes of P minus the number of zero modes of $P^\dagger$. But zero modes of P correspond to (globally defined) symmetries of the Riemann surface, whereas zero modes of $P^\dagger$ are insensitive to coordinate and Weyl transformations, i.e. they correspond to non-trivial metric deformations or moduli of the Riemann surface.

The anomaly of the ghost number current can be calculated with standard methods of QFT [see (1.81)],

$$\partial_{\bar z}j_z=\frac{-3}{8\pi}R. \tag{1.112}$$

Since the integral over the curvature is known to yield the Euler characteristic of a Riemann surface,

$$\frac{1}{4\pi}\int d^2z\sqrt{-g}\,R=2(1-g)=\chi, \tag{1.113}$$

the total violation of ghost number is $6(g-1)$ on a Riemann surface (RS) of genus g (on the sphere this is exactly cancelled by inserting operators with $N_{gh}=3$ times an anti-holomorphic factor with the same ghost number).

The integrated anomaly in N_{gh} is a topological invariant, which, in turn, is proportional to the difference of the number of zero modes of an operator and of its adjoint. Relations of this kind are known as index theorems. Here we obtained a special case of the Riemann-Roch theorem: The number of complex moduli minus the number of complex symmetries of a Riemann surface is $3(g-1)$ (index theorems for other spins can be obtained from the respective formulas for the anomaly; cf. eq. (1.81)).

This relation can be understood directly by building a general RS from a sphere with $2g$ holes: In order to increase the genus by one we have to add two holes, which then are connected by a tube. The positions of the holes add two complex moduli, and the length and the twist angle together count as a third complex modulus. The only exceptional cases are genus 0 and genus 1: The first two holes on the sphere do not generate moduli since their positions can be shifted by $SL(2,\mathbf{C})$ transformations. Hence the torus has only one modulus and one symmetry is left over (if we descrbe the torus as the complex plane modulo the lattice generated by the complex numbers 1 and τ, then τ is the modulus and the symmetry is the translation symmetry of the plane). For genus $g>1$ we have used up all symmetries of the sphere, so there are no more symmetries and the number of moduli is $3g-3$.

So far we only considered the local structure of the moduli space of RSs, which itself can be considered as a complex manifold. The global structure is quite complicated, so we only discuss the simplest case $g=1$. By a rotation and scaling of the lattice we may assume that $\operatorname{Im}\tau>0$. The upper half plane thus parametrizes all inequivalent tori; this space is called Teichmüller space. There are, however, infinitely many different values of τ that parametrize the same torus. Two examples are given by the transformation T, which sends $\tau\to\tau+1$, and S, which sends $\tau\to-1/\tau$. These transformation generate the infinite discrete group $PSL(2,\mathbf{Z})$,

$$\tau\to\frac{a\tau+b}{c\tau+d},\qquad a,b,c,d\in\mathbf{Z},\quad ad-bc=1, \tag{1.114}$$

which is called the modular group. It can be shown that all conformally equivalent tori are related by (1.114). The moduli space, therefore, is the quotient of the Teichmüller space by the modular group. Teichmüller space, in turn, is the universal covering space of moduli space (it does not identify tori that are related by 'big' reparametrizations in $\mathit{Diff}-\mathit{Diff}_0$, i.e. diffeomorphisms that are not continously connected to the identity).

Chapter 2

Supersymmetry

So far we formulated a theory with very nice properties: It describes the interaction of gravitons and, after compactification to four dimensions, gauge bosons. The only trouble is that it is inconsistent: The presense of the tachyon in the spectrum makes the ground state unstable and, even worse, it leads to a divergence of the integral over the modular parameter at genus 1.

A second problem of the bosonic string is that it only describes space-time bosons. So it is natural to introduce additional fields on the world sheet. We may expect that the presense of fermions makes the theory less divergent and hopefully consistent. In fact, there are two approaches to include fermions: We can try to introduce fields that transform as spinors $\theta^A(\sigma,\tau)$ in target space. This is known as the Green–Schwarz (GS) superstring. Indeed, space-time spinors is what we need at the end of the day. But the GS string is difficult to quantize unless we are willing to work in the light-cone gauge.

In order to keep manifest Lorentz invariance in target space we will use the RNS formulation. Here the additional fields $\psi^\mu(\sigma,\tau)$ are spinors on the world sheet and transform as a vector in target space. Decoupling of the negative norm states associated with the time-like component of ψ requires additional constraints on the physical Hilbert space, and hence additional local symmetries of the action. Two such models, the Ramond model [ra71] and the Neveu–Schwarz model [ne71], were constructed already in the early 70s. Later it turned out that the constraints of both models can be derived from the same supersymmetric action [br76, de76]; only the boundary conditions on the fermions are different. For closed strings space-time fermions can arise only if the Ramond model and the NS model are combined. A certain combination is also required by modular invariance, unless we do without space-time fermions. This particular combination, in fact, eliminates the tachyon and results in a space-time supersymmetric theory [gl76]. Hence, consistency of string theory and the presense of fermions (and supersymmetry) seem to be closely related.

The supersymmetric extension of the Polyakov action can be constructed in a number of different ways. Since supersymmetry transforms bosons into fermions we need the superpartner to all bosonic fields: For the scalar target space coordinates X^μ these are just D spin-1/2 fields ψ^μ. An invariant action for spinors requires the introduction of a zweibein field ${e_m}^a$, which defines an orthonormal basis ${E_a}^m\partial_m$ of tanget space in terms of its inverse ${E_a}^m$, i.e. $g^{mn}{e_m}^a{e_n}^b=\eta^{ab}$ with ${E_a}^m{e_m}^b=\delta_a^b$. The additional degree of freedom that has been introduced in this way is not observable since it can be gauged away in an action that is invariant under local Lorentz transformations. The supersymmetric partner of the vielbein is a Rarita–Schwinger field with spin 3/2, the gravitino χ_a (this is a two-component spinor with an additional vector index).

With this field content we can write down the following extension of the Polyakov action:

$$S\sim\int d^2\sigma\sqrt{-g}\,\eta_{\mu\nu}\left(g^{mn}\partial_mX^\mu\partial_nX^\nu+2i\overline{\psi}^\mu\gamma^a{E_a}^mD_m\psi^\nu-i\overline{\chi}_a\gamma^b\gamma^a\psi^\mu({E_b}^n\partial_nX^\nu-\tfrac{i}{4}\overline{\chi}_b\psi^\nu)\right). \tag{2.1}$$

Here γ^a are the two-dimensional γ matrices

$$\gamma^0=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad\gamma^1=\begin{pmatrix}0&i\\i&0\end{pmatrix},\qquad\overline{\gamma}=\gamma^0\gamma^1=\begin{pmatrix}1&0\\0&-1\end{pmatrix}=-\gamma^1\gamma^0 \tag{2.2}$$

in a Majorana-Weyl representation and ψ and χ_a are real spinors (note that the Minkowski signature is essential for the existence of a Majorana representation in two dimensions). D_m is the covariant derivative, so the action is manifestly invariant under general coordinate and local Lorentz transformations, as well as Weyl transformations with weights $(0,-1/2,1,1/2)$ for $(X^\mu,\psi^\mu,{e_m}^a,\chi_m)$. It is also invariant under the local supersymmetry (or supergravity) transformation

$$\delta_\varepsilon X^\mu=i\overline{\varepsilon}\psi^\mu,\qquad\delta_\varepsilon\psi^\mu=\tfrac12\gamma^a({E_a}^m\partial_mX^\mu-\tfrac{i}{2}\overline{\chi}_a\psi^\mu)\varepsilon, \tag{2.3}$$

$$\delta_\varepsilon{e_m}^a=\tfrac{i}{2}\overline{\varepsilon}\gamma^a\chi_m,\qquad\delta_\varepsilon\chi_m=2D_m\varepsilon. \tag{2.4}$$

Calculating the anti-commutator $\{\delta_\varepsilon,\delta_{\varepsilon'}\}$ acting on X^μ we find a contribution that acts like a local translation operator $(\overline{\varepsilon}\gamma^a\varepsilon'){E_a}^m\partial_m$. This is the essential property of a local supersymmmetry algebra. Furthermore, the action (2.1) is also invariant under super-Weyl transformations

$$\delta_\eta\chi_a=\gamma_a\eta,\qquad\delta_\eta X^\mu=\delta_\eta\psi^\mu=\delta_\eta{e_m}^a=0. \tag{2.5}$$

Counting gauge degrees of freedom we observe that there are just enough bosonic symmetries to gauge away the vielbein and enough local fermionic symmetries to gauge away the gravitino. Therefore we can choose the so-called superconformal gauge ${e_m}^a=\delta_m^a$, $\chi_m=0$. Like in the case of the bosonic string the equations of motion of these fields then have to be imposed as constraints on physical states, $T_{mn}\sim\eta_{ab}{e_m}^b\delta S/\delta{e_n}^a=0$ and $T_F^m\sim\delta S/\delta\chi_m=0$, which again can be implemented by using the BRST quantization procedure.

Splitting spinors into their chiral components $\psi = \binom{\psi_+}{\psi_-}$ we obtain the gauge-fixed action

$$S = \tfrac{1}{2\pi} \int d^2\sigma \, (\partial_+ X \partial_- X + i\psi_+ \partial_- \psi_+ + i\psi_- \partial_+ \psi_-) \,, \tag{2.6}$$

which is invariant under supersymmetry transformations (2.3) with $\partial_+ \varepsilon^- = \partial_- \varepsilon^+ = 0$.

The non-vanishing components of the energy momentum tensor T and of its superpartner T_F become

$$T_{\pm\pm} = \tfrac{1}{2} \partial_\pm X \partial_\pm X + \tfrac{i}{2} \psi_\pm \partial_\pm \psi_\pm, \qquad T_{F\pm} = \tfrac{1}{2} \psi_\pm \partial_\pm X, \tag{2.7}$$

so that the theory again splits into left- and right-moving sectors.

2.1 The RNS model

An essential new feature of fermionic strings is that we have some freedom in choosing the boundary conditions of fermions: When we go once around a closed string, observable quantities like correlation functions should not change. Such quantities, however, always contain an even number of spinors. Therefore the fermions ψ^μ may obey periodic or anti-periodic boundary conditions

$$\psi(\sigma + 2\pi) = \pm\psi(\sigma) =: e^{2\pi i \phi} \psi(\sigma) \tag{2.8}$$

with $\phi = 0$ for the Ramond (R) sector and $\phi = 1/2$ for the NS sector. Since left and right movers do not couple we have, in fact, four sectors, namely (R,R), (R,NS), (NS,R) and (NS,NS). A priory, non of the two boundary conditions can be considered to be more natural since the fields ψ have half-integral conformal weight, so if they are periodic on the cylinder they have a cut in the complex plane (R sector). Analytic fields on the punctured plane, on the other hand, have anti-periodic boundary conditions on the cylinder (NS sector).

The mode expansion of the general solution of the equations of motion is

$$\begin{aligned} \psi_+^\mu(\sigma,\tau) &= \sum_{r \in \mathbf{Z}+\phi} b_r^\mu e^{-ir\sigma^+}, \\ \psi_-^\mu(\sigma,\tau) &= \sum_{r \in \mathbf{Z}+\phi} \overline{b}_r^\mu e^{-ir\sigma^-}, \end{aligned} \qquad \begin{cases} \phi = 0 & \text{(R)} \\ \phi = \frac{1}{2} & \text{(NS)} \end{cases} \tag{2.9}$$

and the anti-commutation relations for the quantized oscillators become

$$\{b_r^\mu, b_s^\nu\} \;=\; \eta^{\mu\nu} \delta_{r+s,0} \;=\; \{\overline{b}_r^\mu, \overline{b}_s^\nu\} \tag{2.10}$$

Note that all oscillators transform as target-space vector, so it seems that we will never get the space-time fermions that were our original motivation. In the R sector, however, there are zero modes which form a representation of the Clifford algebra:

$$\{b_0^\mu, b_0^\nu\} \;=\; \eta^{\mu\nu} \;=\; \{\overline{b}_0^\mu, \overline{b}_0^\nu\} \tag{2.11}$$

Therefore the vacuum is degenerate and the various states transform as a target-space spinor! The 'true' vacuum of the conformal field theory turns out to be in the NS sector, and the degenerate Ramond vacua can be obtain from them by a (so far formal) application of a so-called spin field. As the tensor product of two spinor representations only contains representations with integral spin, space-time fermions must arise from the sectors (R,NS) and (NS,R) with mixed boundary conditions.

What we have done so far is to simply add by hand the two Hilbert spaces that provide representations for the oscillator algebras in the two sector of the RNS model. It is not clear if such a procedure is consistent. Indeed, if we want to formulate the model on the torus, which we have to do if we want a theory of interacting strings, then we must do this in a modular invariant way. But it is easy to see that boundary conditions along the two homology cycles of the torus mix under modular transformations (only the completely periodic spin structure is invariant).[1] So if we want to have a Ramond sector then we a forced to also sum over the different boundary conditions in the 'time' directions. This amounts to a projection of the total Hilbert space to states that are even under a certain operator, known as the (world sheet) 'fermion number'. This projection is known as the GSO projection [gl76]. It eventually eliminates the tachyon from the spectrum and makes the theory space-time supersymmetric.

Since we want to have sectors with mixed boundary conditions (R,NS) and (NS,R) the considerations of the last paragraph apply to the left movers and to the right movers seperately, i.e. we have to sum independently over all spin structures of left mover and right movers. The same conlusion applies to higher genera, where the sum extends over all 2^{2g} different spin structures.

Eventually we should write down the super-Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{\hat{c}}{8}m(m^2-2a)\delta_{m+n,0}, \tag{2.12}$$

$$[L_m, G_r] = (\tfrac{m}{2}-r)G_{m+r}, \qquad \{G_r, G_s\} = 2L_{r+s} + \tfrac{\hat{c}}{2}(r^2-\tfrac{a}{2})\delta_{r+s,0}, \tag{2.13}$$

which is the algebra of the Fourier modes

$$L_m = \frac{1}{2}\sum_{n\in\mathbf{Z}} :\alpha_n\alpha_{m-n}: + \frac{1}{2}\sum_{r\in\mathbf{Z}+\phi}(\frac{m}{2}-r):b_r b_{m-r}:, \qquad G_r = \sum_{n\in\mathbf{Z}}\alpha_n b_{r-n} \tag{2.14}$$

of the constraints T and T_F. The algebra for the R sector ($a=0$) and for the NS sector ($a=1/2$) agree formally except for the linear term in the anomaly, which can be shifted by a change in the normal ordering constant in L_0. Indeed, if we let $L_0^R \to L_0^R + \hat{c}/16$ then we obtain (2.12,2.13) with $a=1/2$ in both sectors.

[1] If we parametrize $z=\xi_1+\tau\xi_2$ and assign boundary conditions, or 'spin structures', $(\sigma_1,\sigma_2)=(\pm,\pm)$ with $\psi(\xi_1+1,\xi_2)=\sigma_1\psi(\xi_1,\xi_2)$ and $\psi(\xi_1,\xi_2+1)=\sigma_2\psi(\xi_1,\xi_2)$ then $S:\tau\to-1/\tau$ leaves $(+,+)$ and $(-,-)$ invariant and exchanges $(+,-)$ and $(-,+)$. The other $SL(2,\mathbf{Z})$ generator $T:\tau\to\tau+1$ leaves $(+,\pm)$ invariant and exchanges $(-,-)$ and $(-,+)$.

2.2 Superconformal field theory

In superconformal gauge the vielbein ${e_m}^a = {\delta_m}^a$ is automatically supersymmetric, while a vanishing gravitino ${\chi_m}^\alpha = 0$ requires a compensating super-Weyl transformation, which restricts the supersymmetry parameter to $\partial_\mp\varepsilon^\pm = 0$. Without going through the whole gauge fixing procedure the BRST operator and the ghost action can be reconstructed directly from the algebra of the constraints [ba77]. The resulting Euclidean action can be written as a superspace integral [fr86]

$$S = \tfrac{1}{2}\int d^2z\, d^2\theta\; \overline{D}\mathbf{X}\cdot D\mathbf{X} + \int d^2z\, d^2\theta\; B_{z\theta}\overline{D}C^z + c.c. \tag{2.15}$$

with fermionic coordinates θ, $\overline{\theta}$ and with $\int d\theta := 0$, $\int d\theta\;\theta := 1$, $\partial_\theta := \partial/\partial\theta = \int d\theta$,

$$D = \partial_\theta + \theta\partial, \qquad D^2 = \partial, \qquad\qquad \overline{D} = \partial_{\overline{\theta}} + \overline{\theta}\overline{\partial}, \qquad \overline{D}^2 = \overline{\partial}. \tag{2.16}$$

The superfields $\mathbf{X}^\mu(z,\overline{z},\theta,\overline{\theta})$, $B(z,\theta)$ and $C(z,\theta)$ and their complex conjugates $\overline{B}(\overline{z},\overline{\theta})$ and $\overline{C}(\overline{z},\overline{\theta})$ have superfield expansions

$$\mathbf{X}(z,\overline{z},\theta,\overline{\theta}) = X(z) + X(\overline{z}) + \theta\psi(z) + \overline{\theta\psi}(z), \tag{2.17}$$

$$B(z,\theta) = \beta(z) + \theta b(z), \qquad C(z,\theta) = c(z) + \theta\gamma(z). \tag{2.18}$$

The constraints can also be assembled into a super energy–momentum tensor

$$T(z,\theta) = T_F(z) + \theta T(z) = T_X(z,\theta) + T_{gh}(z,\theta), \tag{2.19}$$

$$T_{(X)}(z,\theta) = -\tfrac{1}{2}D\mathbf{X}\cdot D^2\mathbf{X}, \qquad T_{gh}(z,\theta) = -C(D^2B) + \tfrac{1}{2}(DC)(DB) - \tfrac{3}{2}(D^2C)B. \tag{2.20}$$

$$Q_{BRST} = \oint \tfrac{dz\,d\theta}{2\pi i} j_{BRST} = \oint \tfrac{dz\,d\theta}{2\pi i}(CT_{(X)} - \delta C\, B), \qquad j_{BRST} = C\left(T_{(X)} + \tfrac{1}{2}T_{gh}\right) - \tfrac{3}{4}D\left(C(DC)B\right). \tag{2.21}$$

with $\delta C = C\partial C - \frac{1}{2}(DC)(DC)$. It can be checked that $Q^2_{BRST} = 0$ in 10 dimensions. The OPEs of the superfields at positions (z_1,θ_1) and (z_2,θ_2) can be written in terms of the 'supersymmetric' coordinate displacements $z_{12} = z_1 - z_2 + \theta_1\theta_2$ and $\theta_{12} = \theta_1 - \theta_2$, which satisfy $D_1 z_{12} = \theta_{12}$ and $D_1\theta_{12} = 1$ [fr86]. The superconformal algebra is encoded in

$$T(z_1,\theta_1)T(z_2,\theta_2) \sim \frac{\frac{1}{4}\hat{c}}{z_{12}^3} + \frac{\frac{3}{2}\theta_{12}}{z_{12}^2}T(z_2,\theta_2) + \frac{\frac{1}{2}}{z_{12}}DT(z_2,\theta_2) + \frac{\theta_{12}}{z_{12}}\partial T(z_2,\theta_2) \tag{2.22}$$

with $c = \frac{3}{2}\hat{c}$ and for superfields of superconformal weight h the OPE

$$T(z_1,\theta_1)\Phi(z_2,\theta_2) \sim h\frac{\theta_{12}}{z_{12}^2}\Phi(z_2,\theta_2) + \frac{\frac{1}{2}}{z_{12}}D\Phi(z_2,\theta_2) + \frac{\theta_{12}}{z_{12}}\partial\Phi(z_2,\theta_2) \tag{2.23}$$

follows from the tensorial transformation with a super Lie derivative

$$\delta\Phi = \mathcal{L}_V\Phi = (V\partial + \frac{1}{2}(DV)D + h\partial V)\Phi, \qquad [\mathcal{L}_V, \mathcal{L}_W] = \mathcal{L}_{[V,W]} \tag{2.24}$$

where $V(z,\theta) = v_0 + \theta v_1$ and $[V,W] = V\partial W - W\partial V + \frac{1}{2}DV\,DW$. Taylor expansion and Cauchy's formula read

$$f(z_1,\theta_1) = \sum \frac{1}{n!} z_{12}^n \partial^n (f(z_2,\theta_2) + \theta_{12} Df(z_2,\theta_2)) \tag{2.25}$$

$$\oint \frac{dz_1 d\theta_1}{2\pi i} f(z_1,\theta_1) \frac{1}{z_{12}^{n+1}} = \frac{1}{n!}\partial^n f(z_2,\theta_2), \quad \oint \frac{dz_1 d\theta_1}{2\pi i} f(z_1,\theta_1) \frac{\theta_{12}}{z_{12}^{n+1}} = \frac{1}{n!}\partial^n Df(z_2,\theta_2). \tag{2.26}$$

so that $\mathcal{L}_V \Phi = \oint \frac{dz\,d\theta}{2\pi i} VT\Phi$.

The ferionic components of the conformal superfields are allowed to be double valued, so that we get two different realizations of WS supersymmetry: In the NS sector $(G_{-\frac{1}{2}})^2 = L_{-1}$ gives the translation operator on the complex plane. In the R sector $G_0^2 = L_0 - \frac{1}{16}\hat{c}$ generates translations on the cylinder.

The superfields generate all states in the NS sector from the vacuum. In order to get the Ramond states we can introduce spin fields, which come in pairs $S^\pm(z)$ with

$$|h^\pm\rangle = S^\pm(0)|0\rangle, \qquad G_0|h^\pm\rangle = a_\pm|h^\mp\rangle, \quad a_+ = 1, \quad a_- = h - \tfrac{\hat{c}}{16}. \tag{2.27}$$

In terms of OPEs we thus find

$$T_F(z)S^\pm(w) \sim \frac{1}{2}\frac{a_\pm S^\mp(w)}{(z-w)^{3/2}}. \tag{2.28}$$

$G_0^\dagger = G_0$ implies that $G_0^2 \geq 0$ for all expectation values in a unitary theory. Global SUSY is unbroken in the Ramond sector if $h = \frac{\hat{c}}{16}$ for the ground states. Then we can drop the states $|h^-\rangle$ among the Ramond vacua.

The OPE algebra of the currents $j^{\mu\nu}(z) = \psi^\mu\psi^\nu(z)$ is an SO(10) affine Lie algeba (with level $k = 1$ whose 0-modes (the conserved charges) are the generators of Lorentz transformations on the fermions

$$j^{\mu\nu}(z)\psi^\lambda(w) \sim \frac{1}{z-w}(\eta^{\mu\lambda}\psi^\nu(w) - \eta^{\nu\lambda}\psi^\mu(w)) \tag{2.29}$$

This can be used to derive the OPEs of the spin field because

$$j^{\mu\nu}(z)S(w) \sim \frac{1}{z-w}\gamma^{[\mu}\gamma^{\nu]}S(w) \tag{2.30}$$

For the chiral and antichiral Weyl spinors S_α and S^β, with $h = \frac{15}{24} = \frac{5}{8}$, we obtain

$$\psi^\mu(z)S_\alpha(w) \sim (z-w)^{-\frac{1}{2}}\gamma^\mu_{\alpha\beta}S^\beta \tag{2.31}$$

$$S^\alpha(z)S_\beta(w) \sim (z-w)^{-\frac{5}{4}}\delta^\alpha_\beta + (z-w)^{-\frac{1}{4}}(\tfrac{1}{2}\gamma^\mu\gamma^n)^\beta_\alpha\psi_\mu\psi_\nu \tag{2.32}$$

$$S_\alpha(z)S_\beta(w) \sim (z-w)^{-\frac{3}{4}}\gamma^\mu_{\alpha\beta}\psi_\mu \tag{2.33}$$

None of these OPEs are local. Therefore the fermion vertex operators will require further contributions, which will have to come from (the spin fields for) the ghosts.

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